

# ZONOTOPAL ALGEBRA AND FORWARD EXCHANGE MATROIDS

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**ABSTRACT.** Zonotopal algebra is the study of a family of pairs of dual vector spaces of multivariate polynomials that can be associated with a list of vectors  $X$ . It connects objects from combinatorics, geometry, and approximation theory. The origin of zonotopal algebra is the pair  $(\mathcal{D}(X), \mathcal{P}(X))$ , where  $\mathcal{D}(X)$  denotes the Dahmen-Micchelli space that is spanned by the local pieces of the box spline and  $\mathcal{P}(X)$  is a space spanned by products of linear forms.

The first main result of this paper is the construction of a canonical basis for  $\mathcal{D}(X)$ . We show that it is dual to the canonical basis for  $\mathcal{P}(X)$  that is already known.

The second main result of this paper is the construction of a new family of zonotopal spaces that is far more general than the ones that were recently studied by Ardila-Postnikov, Holtz-Ron, Holtz-Ron-Xu, Li-Ron, and others. We call the underlying combinatorial structure of those spaces forward exchange matroid. A forward exchange matroid is an ordered matroid together with a subset of its set of bases that satisfies a weak version of the basis exchange axiom.

## 1. INTRODUCTION

A finite list of vectors  $X$  gives rise to a large number of objects in combinatorics, algebraic and discrete geometry, commutative algebra, and approximation theory. Examples include matroids, hyperplane arrangements and zonotopes, fat point ideals, and box splines. In the 1980s, various authors in the approximation theory community started studying algebraic structures that capture information about splines (*e.g.* [1, 12, 21]). One important example is the Dahmen-Micchelli space  $\mathcal{D}(X)$ , that is spanned by the local pieces of the box spline and their partial derivatives. See [28, Section 1.2] for a historic survey and the book [13] for a treatment of polynomial spaces appearing in the theory of box splines. Related results were obtained independently by authors interested in hyperplane arrangements (*e.g.* [36]).

The space  $\mathcal{P}(X)$  that is dual to  $\mathcal{D}(X)$  was introduced in [1, 21]. It is spanned by products of linear forms and it can be written as the Macaulay inverse system (or kernel) of an ideal generated by powers of linear forms [11]. Ideals of this type and their inverse systems are also studied in the literature on fat point ideals [24, 25].

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In addition to the aforementioned pair of spaces  $(\mathcal{D}(X), \mathcal{P}(X))$ , Olga Holtz and Amos Ron introduced two more pairs of spaces with interesting combinatorial properties [28]. They named the theory of those spaces *Zonotopal Algebra*. This name reflects the fact that there are various connections between zonotopal spaces and the lattice points in the zonotope defined by  $X$  if the list  $X$  is totally unimodular.

Subsequently, those results were further generalised by Olga Holtz, Amos Ron, and Zhiqiang Xu [29] as well as Nan Li and Amos Ron [33]. Federico Ardila and Alex Postnikov studied generalised  $\mathcal{P}$ -spaces and connections with power ideals [2]. Bernd Sturmfels and Zhiqiang Xu established a connection with Cox rings [41]. Further work on spaces of  $\mathcal{P}$ -type includes [3, 5, 32, 42].

Zonotopal algebra is closely related to matroid theory: the Hilbert series of zonotopal spaces only depend on the matroid structure of the list  $X$ .

It is known that there is a canonical way to construct bases for the spaces of  $\mathcal{P}$ -type [2, 21, 28, 33]. The first of the two main results in this paper is an algorithm that constructs a basis for spaces of  $\mathcal{D}$ -type. Two different algorithms are already known [9, 15]. However, our algorithm has several advantages over the other two: it is canonical and it yields a basis that is dual to the known basis for the  $\mathcal{P}$ -space. Here, canonical means that the basis that we obtain only depends on the order of the elements in the list  $X$  and not on any further choices.

Our second main result is that far more general pairs of zonotopal spaces with nice properties can be constructed than the ones that were previously known. We define a new combinatorial structure called forward exchange matroid. A forward exchange matroid is an ordered matroid together with a subset of its set of bases that satisfies a weak version of the basis exchange axiom. This is the underlying structure of the generalised zonotopal  $\mathcal{D}$ -spaces and  $\mathcal{P}$ -spaces that we introduce.

All objects mentioned so far are part of what we call the continuous theory. If the list  $X$  lies in a lattice (*e.g.*  $\mathbb{Z}^d$ ), an even wider spectrum of mathematical objects appears. We call this the discrete theory. Every object in the continuous theory has a discrete analogue: vector partition functions correspond to box splines and toric arrangements correspond to hyperplane arrangements. The local pieces of the vector partition function are quasi-polynomials that span the discrete Dahmen-Micchelli space  $\text{DM}(X)$ . Both theories are nicely explained in the recent book by Corrado De Concini and Claudio Procesi [17]. The combinatorics of the discrete case is captured by arithmetic matroids which were very recently introduced by Luca Moci and Michele D’Adderio [8, 35].

Vector partition functions arise for example in representation theory as Kostant partition function, when the list  $X$  is chosen to be the set of positive roots of a simple Lie algebra (*e.g.* [7]). There are also applications to the equivariant index theory of elliptic operators [18, 19, 20].

This paper deals only with the continuous theory. However, the two theories overlap if the list of vectors  $X$  is totally unimodular. We hope that the results in this paper can be transferred to the discrete case in the future.

**1.1. Notation.** Our basic object of study is a list of vectors  $X = (x_1, \dots, x_N)$  that span an  $r$ -dimensional real vector space  $U \cong \mathbb{R}^r$ . The dual space  $U^*$  is denoted by  $V$ . We slightly abuse notation by using the symbol  $\subseteq$  for sublists. For  $Y \subseteq X$ ,  $X \setminus Y$  denotes the deletion of a sublist, *i.e.*  $(x_1, x_2) \setminus (x_1) = (x_2)$  even if  $x_1 = x_2$ . The list  $X$  comes with a natural ordering: we say that  $x_i < x_j$  if and only if  $i < j$ .

Note that  $X$  can be identified with a linear map  $\mathbb{R}^N \rightarrow U$  and after the choice of a basis with an  $(r \times N)$ -matrix with real entries.

We will consider families of pairs of dual spaces  $(\mathcal{D}(X, \cdot), \mathcal{P}(X, \cdot))$ . The space  $\mathcal{D}(X, \cdot)$  is contained in  $\text{Sym}(V)$ , the symmetric algebra over  $V$  and  $\mathcal{P}(X, \cdot)$  is contained in  $\text{Sym}(U)$ . The *symmetric algebra* is a base-free version of the ring of polynomials over a vector space. We fix a basis  $(s_1, \dots, s_r)$  for  $U$  and  $(t_1, \dots, t_r)$  denotes the dual basis for  $V$ , *i. e.*  $t_i(s_j) = \delta_{ij}$ , where  $\delta_{ij}$  denotes the Kronecker delta. The choice of the basis determines isomorphisms  $\text{Sym}(U) \cong \mathbb{R}[s_1, \dots, s_r]$  and  $\text{Sym}(V) \cong \mathbb{R}[t_1, \dots, t_r]$ . For  $x \in U$ ,  $x^\circ \subseteq V$  denotes the *annihilator* of  $x$ , *i. e.*  $x^\circ := \{f \in V : f(x) = 0\}$ . For more background on algebra, see [17] or [22].

As usual,  $\chi_S : S \rightarrow \{0, 1\}$  denotes the indicator function of a set  $S$ .

**1.2. Matroids.** An *ordered matroid* on  $N$  elements is a pair  $(A, \mathbb{B})$  where  $A$  is an (ordered) list with  $N$  elements and  $\mathbb{B}$  is a non-empty set of sublists of  $A$  that satisfies the following axiom:

$$\text{Let } B, B' \in \mathbb{B} \text{ and } b \in B \setminus B'. \text{ Then, there exists } b' \in B' \setminus B \text{ s. t. } (B \setminus b) \cup b' \in \mathbb{B} \quad (1.1)$$

$\mathbb{B}$  is called the *set of bases* of the matroid  $(A, \mathbb{B})$ . One can easily show that all elements of  $\mathbb{B}$  have the same cardinality. This number  $r$  is called the *rank* of the matroid  $(A, \mathbb{B})$ . A set  $I \subseteq A$  is called *independent* if it is a subset of a basis. The *rank* of  $Y \subseteq A$  is defined as the cardinality of a maximal independent set contained in  $Y$ . It is denoted  $\text{rk}(Y)$ . The *closure* of  $Y$  is defined as  $\text{cl}(Y) := \{x \in A : \text{rk}(Y \cup x) = \text{rk}(Y)\}$ . A set  $C \subseteq X$  is called a *flat* if  $C = \text{cl}(C)$ .

A set  $C \subseteq X$  is called a *cocircuit* if  $C \cap B \neq \emptyset$  for all bases  $B \in \mathbb{B}$  and  $C$  is minimal with this property. Cocircuits of cardinality one are called *coloops*. An element that is not contained in any basis is called a *loop*.

Now fix a basis  $B \in \mathbb{B}$ . An element  $b \in B$  is called *internally active* in  $B$  if  $b = \max(A \setminus \text{cl}(B \setminus b))$ , *i. e.*  $b$  is the maximal element of the unique cocircuit contained in  $(A \setminus B) \cup b$ . The set of internally active elements in  $B$  is denoted  $I(B)$ . An element  $x \in A \setminus B$  is called *externally active* if  $x \in \text{cl}\{b \in B : b \leq x\}$ , *i. e.*  $x$  is the maximal element of the unique circuit contained in  $B \cup x$ . The set of externally active elements with respect to  $B$  is denoted  $E(B)$ .<sup>1</sup> The *Tutte polynomial*

$$T_{(A, \mathbb{B})}(x, y) := \sum_{B \in \mathbb{B}} x^{|I(B)|} y^{|E(B)|} \quad (1.2)$$

captures a lot of information about the matroid  $(A, \mathbb{B})$ . One can show that it is independent of the order of the list  $A$ .

In this paper, we mainly consider matroids that are realisable over some field  $\mathbb{K}$ . Let  $X = (x_1, \dots, x_N)$  be a list of vectors spanning some  $\mathbb{K}$ -vector space  $W$  and let  $\mathbb{B}(X)$  denote the set of bases for  $W$  that can be selected from  $X$ . One can easily see that  $(X, \mathbb{B}(X))$  is a matroid. The list  $X$  is called a *realisation* of this matroid and a matroid  $(A, \mathbb{B})$  is called *realisable* if there is a list of vectors  $X$  and a bijection between  $A$  and  $X$  that induces a bijection between  $\mathbb{B}$  and  $\mathbb{B}(X)$ .

A standard reference for matroid theory is Oxley's book [37]. Survey papers on the Tutte polynomial are [6, 23].

Throughout this paper, we use a running example, which we now introduce.

<sup>1</sup>Usually, combinatorialists use  $\min$  instead of  $\max$  in the definition of the activities. In the zonotopal algebra literature  $\max$  is used. This has some notational advantages.

**Example 1.1.**

$$\text{Let } X := \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = (x_1, x_2, x_3). \quad (1.3)$$

The set of bases that can be selected from  $X$  is  $\mathbb{B}(X) = \{(x_1, x_2), (x_1, x_3), (x_2, x_3)\}$ . The Tutte polynomial of the matroid  $(X, \mathbb{B}(X))$  is

$$T_{(X, \mathbb{B}(X))}(x, y) = x^2 + x + y. \quad (1.4)$$

**1.3. Some commutative algebra.** In this subsection, we define some commutative algebra terminology that is used in this paper.

**Definition 1.2** (A pairing between symmetric algebras). We define the following pairing:

$$\langle \cdot, \cdot \rangle : \mathbb{R}[s_1, \dots, s_r] \times \mathbb{R}[t_1, \dots, t_r] \rightarrow \mathbb{R} \quad (1.5)$$

$$\langle p, f \rangle := \left( p \left( \frac{\partial}{\partial t_1}, \dots, \frac{\partial}{\partial t_r} \right) f \right) (0), \quad (1.6)$$

*i. e.* we let  $p$  act on  $f$  as a differential operator and take the constant part of the result.

*Remark 1.3.* One can easily show that the definition of the pairing  $\langle \cdot, \cdot \rangle$  is independent of the choice of the bases for the symmetric algebras  $\text{Sym}(U)$  and  $\text{Sym}(V)$  as long as the bases are dual to each other.

**Definition 1.4.** Let  $\mathcal{I} \subseteq \mathbb{R}[s_1, \dots, s_r]$  be a homogeneous ideal. Its *kernel* or Macaulay inverse system [26, 27, 34] is defined as

$$\ker \mathcal{I} := \{f \in \mathbb{R}[t_1, \dots, t_r] : \langle q, f \rangle = 0 \text{ for all } q \in \mathcal{I}\}. \quad (1.7)$$

*Remark 1.5.*  $\ker \mathcal{I}$  can also be written as

$$\ker \mathcal{I} := \{f \in \mathbb{R}[t_1, \dots, t_r] : p \left( \frac{\partial}{\partial t_1}, \dots, \frac{\partial}{\partial t_r} \right) f = 0\} \quad (1.8)$$

where  $p$  runs over a set of generators for the ideal  $\mathcal{I}$ .

*Remark 1.6.* For a homogeneous ideal  $\mathcal{I} \subseteq \mathbb{R}[s_1, \dots, s_r]$  of finite codimension the Hilbert series of  $\ker \mathcal{I}$  and  $\mathbb{R}[s_1, \dots, s_r]/\mathcal{I}$  are equal. For instance, this follows from [17, Theorem 5.4].

A *graded vector space* is a vector space  $V$  that decomposes into a direct sum  $V = \bigoplus_{i \geq 0} V_i$ . A graded linear map  $f : V \rightarrow W$  preserves the grade, *i. e.*  $f(V_i)$  is contained in  $W_i$ . For a graded vector space, we define its *Hilbert series* as the formal power series  $\text{Hilb}(V, t) := \sum_{i \geq 0} \dim(V_i) t^i$ .

Note that a linear map  $f : V \rightarrow W$  induces an algebra homomorphism  $\text{Sym}(f) : \text{Sym}(V) \rightarrow \text{Sym}(W)$ .

**1.4. Central zonotopal algebra.** In this subsection, we define the Dahmen-Micchelli space  $\mathcal{D}(X)$  and its dual  $\mathcal{P}(X)$ . The pair  $(\mathcal{D}(X), \mathcal{P}(X))$ , which is called the central pair of zonotopal spaces in [28], is the origin of zonotopal algebra.

A vector  $u \in U$  naturally defines a polynomial  $p_u \in \mathbb{R}[s_1, \dots, s_r]$  as follows: if  $u$  can be expressed in the basis  $(s_1, \dots, s_r)$  as  $u = \sum_{i=1}^r \lambda_i s_i$ , then we define  $p_u := \sum_{i=1}^r \lambda_i s_i \in \mathbb{R}[s_1, \dots, s_r]$ . For  $Y \subseteq X$ , we define  $p_Y := \prod_{x \in Y} p_x$ .

**Definition 1.7.** Let  $X \subseteq U \cong \mathbb{R}^r$  be a finite list of vectors that spans  $U$ . Then we define

$$\mathcal{J}(X) := \text{ideal}\{p_T : T \subseteq X \text{ cocircuit}\} \subseteq \mathbb{R}[s_1, \dots, s_r] \quad (1.9)$$

$$\text{and } \mathcal{D}(X) := \ker \mathcal{J}(X) \subseteq \mathbb{R}[t_1, \dots, t_r]. \quad (1.10)$$

$\mathcal{D}(X)$  is called the central  $\mathcal{D}$ -space or *Dahmen-Micchelli* space.

It can be shown that  $\mathcal{D}(X)$  is the space spanned by the local pieces of the box spline and their partial derivatives. The box spline will be defined in the next section. The space  $\mathcal{D}(X)$  was introduced in [12] and in [10] it was shown that its dimension is  $|\mathbb{B}(X)|$ .

**Definition 1.8.** Let  $X \subseteq U \cong \mathbb{R}^r$  be a finite list of vectors that spans  $U$ . Then, we define the central  $\mathcal{P}$ -space

$$\mathcal{P}(X) := \text{span}\{p_Y : Y \subseteq X, X \setminus Y \text{ has full rank}\} \subseteq \mathbb{R}[s_1, \dots, s_r]. \quad (1.11)$$

**Proposition 1.9** ([21]). *Let  $X \subseteq U \cong \mathbb{R}^r$  be a finite list of vectors that spans  $U$ . A basis for  $\mathcal{P}(X)$  is given by*

$$\mathcal{B}(X) := \{Q_B : B \in \mathbb{B}(X)\}, \quad (1.12)$$

where  $Q_B := p_{X \setminus (B \cup E(B))}$ .

The space  $\mathcal{P}(X)$  can also be written as the kernel of an ideal. The following proposition appeared in [2] and earlier in a slightly different version in [11].

**Proposition 1.10.** *Let  $X \subseteq U \cong \mathbb{R}^r$  be a finite list of vectors that spans  $U$ . Then,*

$$\mathcal{P}(X) = \ker \mathcal{I}(X), \quad (1.13)$$

$$\text{where } \mathcal{I}(X) := \text{ideal}\left\{p_\eta^{m(\eta)} : \eta \in V \setminus \{0\}\right\} \subseteq \mathbb{R}[t_1, \dots, t_r] \quad (1.14)$$

and  $m : V \rightarrow \mathbb{N}$  assigns to  $\eta \in V$  the number of vectors in  $X$  that are not perpendicular to  $\eta$ .

**Example 1.11.** Let  $X$  be the list of vectors we defined in Example 1.1. Then

$$\mathcal{D}(X) = \ker \text{ideal}\{s_1 s_2, s_1(s_1 + s_2), s_2(s_1 + s_2)\} = \text{span}\{1, t_1, t_2\}, \quad (1.15)$$

$$\mathcal{I}(X) = \text{ideal}\{t_1^2, t_2^2, (t_1 - t_2)^2\} + \mathbb{R}[t_1, t_2]_{\geq 3} = \text{ideal}\{t_1^2, t_2^2, t_1 t_2\}, \quad (1.16)$$

$$\text{and } \mathcal{P}(X) = \ker \mathcal{I}(X) = \text{span}\{1, s_1, s_2\}. \quad (1.17)$$

**Proposition 1.12** ([21, 30]). *Let  $X \subseteq U \cong \mathbb{R}^r$  be a finite list of vectors that spans  $U$ . Then the spaces  $\mathcal{P}(X)$  and  $\mathcal{D}(X)$  are dual under the pairing  $\langle \cdot, \cdot \rangle$ , i. e.*

$$\mathcal{D}(X) \rightarrow \mathcal{P}(X)^* \quad (1.18)$$

$$f \mapsto \langle \cdot, f \rangle \quad (1.19)$$

is an isomorphism.

The preceding proposition implies that the Hilbert series of  $\mathcal{P}(X)$  and  $\mathcal{D}(X)$  are equal. By Proposition 1.9, this Hilbert series is a matroid invariant and a specialisation of the Tutte polynomial. These facts are summarised in the following proposition.

**Proposition 1.13.** *Let  $X \subseteq U \cong \mathbb{R}^r$  be a list of vectors  $N$  vectors that spans  $U$ . Then*

$$\text{Hilb}(\mathcal{D}(X), q) = \text{Hilb}(\mathcal{P}(X), q) = q^{N-r} T_{(X, \mathbb{B}(X))} \left(1, \frac{1}{q}\right) = \sum_{B \in \mathbb{B}(X)} q^{N-r-|E(B)|}. \quad (1.20)$$

*Remark 1.14.* Most of the results mentioned above also hold over other fields of characteristic zero or even over arbitrary fields. In this paper we work over the reals, because the construction in Section 3 uses distributions. Nevertheless, we believe that some of our results also hold over other fields.

**1.5. Organisation of the article.** The remainder of this article is organised as follows. In Section 2 we provide some additional mathematical background, in particular on splines.

In Section 3 we construct certain polynomials  $R^B$  as convolutions of differences of multivariate splines. In Section 4 we show that

$$B(X) := \{\det(B) R^B : B \in \mathbb{B}(X)\} \quad (1.21)$$

is a basis for  $\mathcal{D}(X)$  and we prove that this basis is dual to the basis  $\mathcal{B}(X)$  for  $\mathcal{P}(X)$ . In Section 5 we discuss deletion-contraction and two short exact sequences. In Section 6 we introduce a new combinatorial structure called forward exchange matroid. This is an ordered matroid together with a subset  $\mathbb{B}'$  of its set of bases with the so-called forward exchange property. In Section 7 we introduce the generalised  $\mathcal{P}$ -space  $\mathcal{P}(X, \mathbb{B}') := \text{span}\{Q_B : B \in \mathbb{B}'\}$  and the generalised  $\mathcal{D}$ -space  $\mathcal{D}(X, \mathbb{B}')$ . We show that most of the results that we described in Subsection 1.4 and Section 4 still hold for these spaces if  $\mathbb{B}'$  has the forward exchange property. For example, the two spaces are dual and a suitable subset of  $B(X)$  turns out to be a basis for  $\mathcal{D}(X, \mathbb{B}')$ . Furthermore,  $\mathcal{D}(X, \mathbb{B}')$  and  $\mathcal{P}(X, \mathbb{B}')$  have deletion-contraction decompositions that are related to the deletion-contraction reduction of the Tutte polynomial.

In Section 8 we review the previously known zonotopal spaces and we show that they are special cases of our spaces  $\mathcal{D}(X, \mathbb{B}')$  and  $\mathcal{P}(X, \mathbb{B}')$ .

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## 2. PRELIMINARIES

In this section we provide some mathematical background. In Subsection 2.1 we define some objects from discrete geometry. In Subsection 2.2 we review distributions and in Subsection 2.3 we discuss box splines and multivariate splines. Subsection 2.4 contains information about previously known algorithms for the construction of bases for  $\mathcal{D}$ -spaces.

### 2.1. Cones and zonotopes.

**Definition 2.1.** Let  $X = (x_1, \dots, x_N) \subseteq U \cong \mathbb{R}^r$  be a list of vectors. Then we define the *zonotope*  $Z(X)$  and the *cone*  $\text{cone}(X)$  by

$$Z(X) := \left\{ \sum_{i=1}^N \lambda_i x_i : 0 \leq \lambda_i \leq 1 \right\} \text{ and } \text{cone}(X) := \left\{ \sum_{i=1}^N \lambda_i x_i : \lambda_i \geq 0 \right\}. \quad (2.1)$$

Zonotopes are closely connected to zonotopal algebra. For example, if  $X$  is totally unimodular, then  $\text{vol}(Z(X)) = \dim \mathcal{P}(X) = \dim \mathcal{D}(X)$  and the space  $\mathcal{P}(X)$  can be obtained from the set of lattice points in the half-open zonotope via least map interpolation that is explained in Subsection 2.4.

**2.2. Distributions.** In this paper we are mainly interested in multivariate polynomials. However, in the construction in Section 3 more general objects appear as intermediate products. In this construction, we need "generalised polynomials" whose support is contained in a subspace. Furthermore, we use convolutions and the fact that convolutions and partial derivatives commute. Distributions have all of the desired properties. In this subsection we summarise important facts about distributions that we will need later on. For a detailed introduction to the subject, we refer the reader to Laurent Schwartz's book [39].

A *distribution* on a vector space  $U \cong \mathbb{R}^r$  (or an open subset of  $U$ ) is a continuous linear functional that maps a test function to a real number. *Test functions* are compactly supported smooth functions  $U \rightarrow \mathbb{R}$ . An important example of a distribution is the delta distribution  $\delta_x$  given by  $\delta_x(\varphi) := \varphi(x)$ . A locally integrable function  $f : U \rightarrow \mathbb{R}$  defines a distribution  $T_f$  in the following way:

$$T_f(\varphi) := \int_U f(u)\varphi(u) du. \quad (2.2)$$

Recall that for two functions  $f, g : U \rightarrow \mathbb{R}$ , the *convolution* is defined as

$$f * g := \int_U f(u)g(\cdot - u) du. \quad (2.3)$$

This is well-defined only if  $f$  and  $g$  decay sufficiently rapidly at infinity in order for the integral to exist. The convolution of two distributions can also be defined under certain conditions.

A distribution  $T$  vanishes on a set  $\Gamma \subseteq U$  if  $T(\varphi) = 0$  for all test functions whose support is contained in  $\Gamma$ . The support  $\text{supp}(T)$  of  $T$  is the complement of the maximal open set on which  $T$  vanishes.

Let  $S_\xi$  and  $T_\eta$  be two distributions for which  $\text{supp}(S_\xi) \cap (K - \text{supp}(T_\eta))$  is compact for any compact set  $K$ . Let  $\varphi : U \rightarrow \mathbb{R}$  be a test function with support  $K$ . Then, we define the convolution

$$(S_\xi * T_\eta)(\varphi) := S_\xi(T_\eta(\alpha(\xi)\varphi(\xi + \eta))), \quad (2.4)$$

where  $\alpha$  is a test function that is equal to one on a neighbourhood of  $\text{supp}(S_\xi) \cap (K - \text{supp}(T_\eta))$ . When evaluating  $T_\eta(\alpha(\xi)\varphi(\xi + \eta))$ , we think of  $\varphi(\xi + \eta)$  as a function in  $\eta$  and of  $\xi$  as a fixed parameter. Then,  $T_\eta(\alpha(\xi)\varphi(\xi + \eta))$  is a function in  $\xi$  with compact support that is contained in  $K - \text{supp}(T_\eta)$ . Note that the definition of  $(S_\xi * T_\eta)(\varphi)$  is independent of the choice of the function  $\alpha$ . The multiplication by  $\alpha$  is necessary to ensure that  $T_\eta(\alpha(\xi)\varphi(\xi + \eta))$  as a function in  $\xi$  has compact support.

Note that the convolution of two distributions is a commutative operation and  $T * \delta_0 = T$ . Let  $u \in U$ . The partial derivative of a distribution  $T$  in direction  $u$  is defined by  $(D_u T)(\varphi) := -T(D_u \varphi)$ . Convolutions of distributions have the same nice property with respect to partial derivatives as convolutions of functions. Namely, if  $T_1$  and  $T_2$  are distributions on  $U$  and  $u \in U$ , then

$$D_u(T_1 * T_2) = (D_u T_1) * T_2 = T_1 * (D_u T_2). \quad (2.5)$$



**2.3. Splines.** In this subsection we introduce multivariate splines and box splines as in [17, Chapter 7]. Another good reference is [13].

**Definition 2.2.** Let  $X \subseteq U \cong \mathbb{R}^r$  be a finite list of vectors. The *multivariate spline* (or truncated power)  $T_X$  and the *box spline*  $B_X$  are distributions that are characterised by the formulae

$$\int_U f(u) B_X(u) du = \int_0^1 \cdots \int_0^1 f\left(\sum_{i=1}^N \lambda_i x_i\right) d\lambda_1 \cdots d\lambda_N \quad (2.6)$$

$$\text{and } \int_U f(u) T_X(u) du = \int_0^\infty \cdots \int_0^\infty f\left(\sum_{i=1}^N \lambda_i x_i\right) d\lambda_1 \cdots d\lambda_N. \quad (2.7)$$

The multivariate spline is well-defined only if  $\text{cone}(X)$  is pointed or equivalently if there is a functional  $\varphi \in V$  s. t.  $\varphi(x) > 0$  for all  $x \in X$ . It is of course always possible to multiply certain entries of the list  $X$  by  $-1$  s. t. this condition is satisfied. Note that in Definition 2.2, we do *not* require that  $X$  spans  $U$  in contrast to most of the rest of this paper.

$B_X$  and  $T_X$  can be identified with the functions

$$B_X(u) = \frac{1}{\sqrt{\det(XX^T)}} \text{vol}_{N-\dim(\text{span}(X))} \{z \in [0; 1]^N : Xz = u\} \quad (2.8)$$

$$\text{and } T_X(u) = \frac{1}{\sqrt{\det(XX^T)}} \text{vol}_{N-\dim(\text{span}(X))} \{z \in \mathbb{R}_{\geq 0}^N : Xz = u\}. \quad (2.9)$$

It follows immediately from (2.8) and (2.9) that  $B_X$  is supported in the zonotope  $Z(X)$  and  $T_X$  is supported in the cone  $\text{cone}(X)$ . For a basis  $C \subseteq U$ ,

$$B_C = \frac{\chi_{Z(C)}}{|\det(C)|} \text{ and } T_C = \frac{\chi_{\text{cone}(C)}}{|\det(C)|}. \quad (2.10)$$

*Remark 2.3.* The box spline can easily be obtained from the multivariate spline. Namely,

$$B_X(x) = \sum_{S \subseteq X} (-1)^{|S|} T_X(x - a_S), \quad (2.11)$$

where  $a_S := \sum_{a \in S} a$ .

From now on, we will only consider  $T_X$ . We introduced the box spline only because of its importance in approximation theory.

**Theorem 2.4.** Let  $X \subseteq U \cong \mathbb{R}^r$  be a finite list of vectors that spans  $U$ . The cone  $\text{cone}(X)$  can be decomposed into finitely many cones  $C_i$  s. t.  $T_X$  restricted to each  $C_i$  is a homogeneous polynomial of degree  $N - r$ .

**Theorem 2.5.** The space spanned by the local pieces of the multivariate spline  $T_X$  and their partial derivatives is equal to the Dahmen-Micchelli space  $\mathcal{D}(X)$  that was defined in Definition 1.7.

The multivariate spline can also be defined inductively by the convolution formula

$$T_{(X,x)} = T_X * T_{(x)} = \int_0^\infty T_X(\cdot - \lambda x) d\lambda \quad (2.12)$$



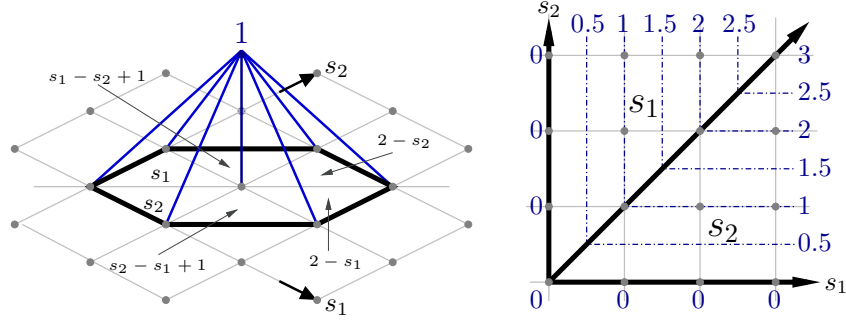


FIGURE 1. The box spline and the multivariate spline defined by the list  $X$  in Example 1.1. The multivariate spline is calculated in Example 2.6. On the right, the dashed lines are level curves.

using (2.10) as a starting point. In particular,  $T_X = T_{x_1} * \dots * T_{x_N}$ . Since  $D_x T_x = \delta_0$ , the convolution formula implies for  $Y \subseteq X$  that

$$D_Y T_X = T_{X \setminus Y} \text{ where } D_Y := \prod_{x \in Y} D_x. \quad (2.13)$$

**Example 2.6.** We consider the same list  $X$  as in Example 1.1. By (2.10),  $T_{(x_1, x_2)}$  is the indicator function of  $\mathbb{R}_{\geq 0}^2$ . Then, by (2.12), we can deduce

$$T_X(s_1, s_2) = \int_0^\infty \chi_{\mathbb{R}_{\geq 0}^2}(s_1 - \lambda, s_2 - \lambda) d\lambda = \min(s_1, s_2). \quad (2.14)$$

See Figure 1 for a graphic description of  $T_X$ .

**Example 2.7.** Let  $X_i := \underbrace{(1, \dots, 1)}_{i \text{ times}}$ . Then,

$$T_{X_1}(s) = \chi_{\mathbb{R}_{\geq 0}}(s) \quad (2.15)$$

$$\text{and } T_{X_{i+1}}(s) = \int_0^\infty T_{X_i}(s - \lambda) d\lambda = \int_0^s \frac{\lambda^i}{i!} d\lambda = \frac{s^{i+1}}{(i+1)!} \text{ for } s \geq 0. \quad (2.16)$$

**2.4. Previously known methods for constructing bases for  $\mathcal{D}$ -spaces.** Two other methods are known to construct a basis for  $\mathcal{D}(X)$ . However, our algorithm has several advantages over the other two: it is canonical, *i. e.* it only depends on the order of the list  $X$  and it yields a basis that is dual to the known basis  $\mathcal{B}(X)$  for the  $\mathcal{P}$ -space.

In Wolfgang Dahmen's construction [9], polynomials are chosen as basis elements that are local pieces of certain multivariate splines. For certain choices of the parameters in his construction, it might yield the same basis as ours.

The second construction uses the so-called least map interpolation that was introduced by Carl de Boor and Amos Ron [14]. Given a finite set  $S \subseteq V$ , they construct a space of polynomials in  $\Pi(S) \subseteq \text{Sym}(V)$  of dimension  $|S|$  with certain nice properties.

Recall that  $U \cong \mathbb{R}^r$ ,  $V$  denotes the dual space and a vector  $v \in V$  defines a linear form  $p_v \in \mathbb{R}[t_1, \dots, t_r] \cong \text{Sym}(V)$ . We define the exponential function as usual by

$$e^v := \sum_{j \geq 0} \frac{p_v^j}{j!} \in \mathbb{R}[[t_1, \dots, t_r]] \cong \text{Sym}(U)^*. \quad (2.17)$$

The least map  $\downarrow$  maps a non-zero element of the ring of formal power series  $\mathbb{R}[[t_1, \dots, t_r]]$  to its homogeneous component of lowest degree that is non-zero. The *least space* of a finite set  $S \subseteq V$  is defined as

$$\Pi(S) := \text{span}\{f_\downarrow : f \in \text{span}\{e^v : v \in S\}\} \subseteq \mathbb{R}[t_1, \dots, t_r]. \quad (2.18)$$

Let  $x \in U$  and  $c_x \in \mathbb{R}$ . This defines a *hyperplane*

$$H_{x, c_x} := \{v \in V : v(x) = c_x\}. \quad (2.19)$$

If we fix a vector  $c \in \mathbb{R}^X$ , we obtain a hyperplane arrangement  $\mathcal{H}(X, c) = \{H_{x, c_x} : x \in X\}$ .

Every basis  $B \subseteq X$  determines a unique vertex  $\theta_B \in V$  of the hyperplane arrangement  $\mathcal{H}(X, c)$  that satisfies  $\theta_B(x) = c_x$  for all  $x \in B$ . In matrix notation,  $\theta_B = B^{-1}c_B$ , where  $c_B$  denotes the restriction of  $c$  to  $\mathbb{R}^B$ . If the vector  $c$  is sufficiently generic, then  $\theta_B \neq \theta_{B'}$  for distinct bases  $B$  and  $B'$ . In this case, the hyperplane arrangement  $\mathcal{H}(X, c)$  is said to be in general position. For more information on hyperplane arrangements, see [40].

The following surprising theorem makes a connection between hyperplane arrangements and the space  $\mathcal{D}(X)$ . It generalises to other  $\mathcal{D}$ -spaces (see [28, 29, 33]).

**Theorem 2.8** ([14]). *Let  $X \subseteq U \cong \mathbb{R}^r$  be a finite list of vectors that spans  $U$ . Let  $c \in \mathbb{R}^X$  be a vector s. t. the hyperplane arrangement  $\mathcal{H}(X, c)$  is in general position and let  $S$  be the set of vertices of  $\mathcal{H}(X, c)$ . Then*

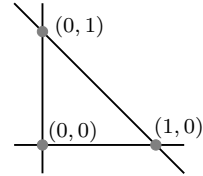
$$\mathcal{D}(X) = \Pi(S). \quad (2.20)$$

De Boer and Ron give a method to select a basis from  $\Pi(S)$  in [15] (see also [13, Chapter II] for a summary). Their construction depends on the choice of the vector  $c$ , an ordering of the bases and an ordering of  $\mathbb{N}^r$  while our construction only depends on the order on  $X$ .

**Example 2.9.** This is a continuation of Example 1.1.

Let  $c_1 = c_2 = 0$  and  $c_3 = 1$ . The set of vertices of  $\mathcal{H}(X, c)$  is  $S = \{(0, 0), (1, 0), (0, 1)\}$ . Then,

$$\begin{aligned} \Pi(S) &= \text{span}\{f_\downarrow : f \in \text{span}\{1, e^{t_1}, e^{t_2}\}\} \\ &= \text{span}\{1, t_1, t_2\}, \text{ since } 1 = 1_\downarrow, \\ t_1 &= (e^{t_1} - 1)_\downarrow, \text{ and } t_2 = (e^{t_2} - 1)_\downarrow. \end{aligned}$$



### 3. CONSTRUCTION OF BASIS ELEMENTS

In this section we construct a polynomial  $R_Z^B$  in  $\mathbb{R}[s_1, \dots, s_r]$ , given a finite list  $Z \subseteq U \cong \mathbb{R}^r$  and a basis  $B \subseteq Z$ . Later on we show that polynomials of this type form bases for various zonotopal  $\mathcal{D}$ -spaces if one chooses suitable pairs  $(B, Z)$ . The polynomial  $R_Z^B$  is constructed as a convolution of differences of multivariate splines.

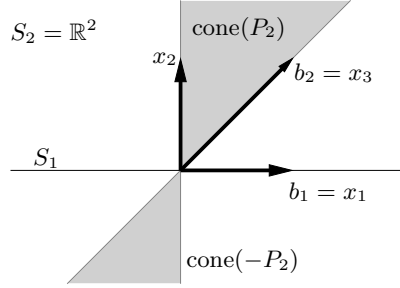


FIGURE 2. The geometry of the construction of the polynomial  $R_X^{(x_1, x_3)}$  in Example 4.4. Note that  $N_2 = \emptyset$ .

Let  $Z \subseteq U$  be a finite list and let  $B = (b_1, \dots, b_r) \subseteq Z$  be a basis. It is important that the basis is ordered and that this order is the order obtained by restricting the order on  $Z$  to  $B$ . For  $i \in \{0, \dots, r\}$ , we define  $S_i = S_i^B := \text{span}\{b_1, \dots, b_i\}$ . Hence,

$$\{0\} = S_0^B \subsetneq S_1^B \subsetneq S_2^B \subsetneq \dots \subsetneq S_r^B = U \cong \mathbb{R}^r \quad (3.1)$$

is a flag of subspaces. We define an orientation on each of the spaces  $S_i$  by saying that  $(b_1, \dots, b_i)$  is a positive basis for  $S_i$ . Now a basis  $D = (d_1, \dots, d_i)$  for  $S_i$  is called positive if the map that sends  $b_\nu$  to  $d_\nu$  for  $1 \leq \nu \leq i$  has positive determinant.

Let  $u \in S_i \setminus S_{i-1}$ . If  $(b_1, \dots, b_{i-1}, u)$  is a positive basis, we call  $u$  positive. Otherwise, we call  $u$  negative. We partition  $Z \cap (S_i \setminus S_{i-1})$  as follows:

$$P_i^B := \{u \in Z \cap (S_i \setminus S_{i-1}) : u \text{ positive}\} \quad (3.2)$$

$$\text{and } N_i^B := \{u \in Z \cap (S_i \setminus S_{i-1}) : u \text{ negative}\}. \quad (3.3)$$

We define

$$T_i^{B+} := (-1)^{|N_i|} \cdot T_{P_i} * T_{-N_i} \text{ and } T_i^{B-} := (-1)^{|P_i|} \cdot T_{-P_i} * T_{N_i}. \quad (3.4)$$

Note that  $T_i^{B+}$  is supported in  $\text{cone}(P_i, -N_i)$  and that

$$T_i^{B-}(x) = (-1)^{|P_i \cup N_i|} T_i^{B+}(-x). \quad (3.5)$$

Now define

$$R_i^B := T_i^{B+} - T_i^{B-} \quad \text{and} \quad R_Z^B = R^B := R_1^B * \dots * R_r^B. \quad (3.6)$$

For an example of this construction see Example 4.4 and Figure 2. In Corollary 3.4, we will see that the distribution  $R_Z^B$  can be identified with a homogeneous polynomial.

*Remark 3.1.* A similar construction of certain quasi-polynomials in the discrete case is done in [17, 13.6].

*Remark 3.2.* The construction of the polynomials  $R_Z^B$  may at first seem rather complicated in comparison with construction of the polynomials  $Q_B$  that form bases of the  $\mathcal{P}$ -spaces.

Here are a few remarks to explain this construction: multivariate splines are very convenient because it is so easy to calculate their partial derivatives (cf. (2.13)). Taking differences of two splines in the definition of  $R_i^B$  ensures that  $R_Z^B$  is a

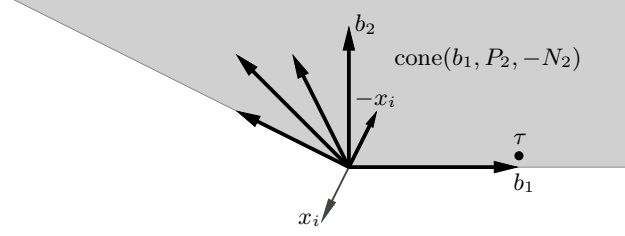


FIGURE 3. The setup in the proof of Proposition 3.3.

polynomial and not just piecewise polynomial. In fact,  $R_1^B * \dots * R_i^B$  is a “polynomial supported in  $S_i$ ” for all  $i$ .

We have to change the sign of some of the vectors before constructing the multivariate spline  $T_i^{B+}$  to ensure that all the convolutions are well-defined. For example, the convolutions in (3.6) are well-defined for the following reason: the support of  $R_1^B * \dots * R_i^B$  is contained in  $S_i$ . The support of  $R_{i+1}^B$  is  $\text{cone}(P_{i+1}, -N_{i+1}) \cup \text{cone}(-P_{i+1}, N_{i+1})$ . For every compact set  $K$ , the set

$$(S_i \cap (K - (\text{cone}(P_{i+1}, -N_{i+1}) \cup \text{cone}(P_{i+1}, -N_{i+1})))) \quad (3.7)$$

is compact.

**Proposition 3.3.** *The distribution  $R_Z^B$  is a local piece of the multivariate spline  $T_1^{B+} * \dots * T_r^{B+}$ .*

*Proof.* Let  $c \gg 0$  and let

$$\tau := b_1 + \frac{1}{c}b_2 + \dots + \frac{1}{c^{r-2}}b_{r-1} + \frac{1}{c^{r-1}}b_r. \quad (3.8)$$

See Figure 3 for an example of this construction. The vector  $\tau$  is contained in  $\text{cone}(Z)$ . By Theorem 2.4, there exists a subcone of  $\text{cone}(Z)$  that contains  $\tau$  s.t.  $T_Z$  agrees with a polynomial  $p_{\tau, Z}$  on this subcone. We claim that  $R_Z^B$  is equal to  $p_{\tau, Z}$ . Note that

$$R_Z^B = (T_1^{B+} - T_1^{B-}) * \dots * (T_r^{B+} - T_r^{B-}) = \sum_{J \subseteq [r]} (-1)^{|J| + \sum_{i \notin J} |N_i| + \sum_{i \in J} |P_i|} T_{Z_B^J},$$

where

$$Z_B^J = \bigcup_{i \notin J} (P_i, -N_i) \cup \bigcup_{i \in J} (-P_i, N_i).$$

In order to prove our claim, it is sufficient to show that  $\tau$  is contained in  $\text{cone}(Z_B^J)$  if and only if  $J = \emptyset$ . The “if” part is clear.

Let  $J$  be non-empty and let  $j^*$  be the minimal element. For  $\alpha \in \mathbb{R}$  let  $\phi_\alpha : U \rightarrow \mathbb{R}$  be the linear form that maps a vector  $x$  to

$$\sum_{j=j^*}^r (-1)^{\chi_J(j)} \alpha^j \lambda_j, \quad (3.9)$$

where  $\lambda_j$  denotes the coefficient of  $b_j$  when  $x$  is written in the basis  $(b_1, \dots, b_r)$ . We claim that for sufficiently large  $\alpha$ ,  $\phi_\alpha$  is non-negative on  $Z_B^J$  and  $\phi_\alpha(\tau) < 0$ .

By Farkas' Lemma (e.g. [38, Section 5.5]), this proves that  $\tau$  is not contained in  $\text{cone}(Z_B^J)$ .

If  $x \in S_i \cap Z_B^J$  for  $i < j^*$ , then obviously  $\phi_\alpha(x) = 0$ . If  $x \in (S_i \setminus S_{i-1}) \cap Z_B^J$  for  $i \geq j^*$ , then  $\lambda_i \neq 0$  and  $\lambda_\nu = 0$  for all  $\nu \geq i+1$ . In addition,  $(-1)^{\chi_{J(i)}} \lambda_i > 0$ , since all vectors in  $(P_i, -N_i)$  have a positive  $b_i$  component when written in the basis  $(b_1, \dots, b_r)$ . Hence,  $\phi_\alpha(x) = (-1)^{\chi_{J(i)}} \alpha^i \lambda_i + o(\alpha^i) = \alpha^i |\lambda_i| + o(\alpha^i)$ . This is positive for sufficiently large  $\alpha$ . For  $\tau$ , we obtain

$$\phi(\tau) = -\frac{\alpha^{j^*}}{c^{j^*-1}} \pm \frac{\alpha^{j^*+1}}{c^{j^*}} \pm \dots = -\frac{\alpha^{j^*}}{c^{j^*-1}} + o\left(\frac{1}{c^{j^*-1}}\right). \quad (3.10)$$

This is negative for sufficiently large  $c$ . Note that we fix a large  $\alpha$  first and then we let  $c$  grow.  $\square$

Using Theorem 2.4, we can deduce the following Corollary.

**Corollary 3.4.** *The distribution  $R_Z^B$  can be identified with a homogeneous polynomial of degree  $|Z| - r$ .*

*Remark 3.5.* The local pieces of the multivariate spline are uniquely determined by a certain equation (cf. [17, Theorems 9.5 and 9.7]). Taking into account Proposition 3.3, this gives us a different method to calculate the polynomials  $R_Z^B$ .

The following theorem that is due to Zhiqiang Xu yields another formula for the polynomials  $R_Z^B$ .

**Theorem 3.6** ([43, Theorem 3.1.]). *Let  $X \subseteq U \cong \mathbb{R}^r$  be a list of  $N$  vectors that spans  $U$ . Let  $c \in \mathbb{R}^X$  be a vector s. t. the hyperplane arrangement  $\mathcal{H}(X, c)$  is in general position. For a basis  $B \in \mathbb{B}(X)$ , let  $\theta_B \in V$  denote the vertex of  $\mathcal{H}(X, c)$  corresponding to  $B$  (cf. Subsection 2.4). Then*

$$T_X(u) = \frac{1}{(N-r)!} \sum_{B \in \mathbb{B}(X)} \frac{(-\theta_B u)^{N-r}}{|\det(B)| \prod_{x \in X \setminus B} (\theta_B x - c_x)} \chi_{\text{cone}(B)}(u). \quad (3.11)$$

Note that the numerator (3.11) is non-zero because  $c$  is chosen s. t.  $\mathcal{H}(X, c)$  is in general position. Using Proposition 3.3, one can deduce the following corollary.

**Corollary 3.7.** *Let  $Z \subseteq U \cong \mathbb{R}^r$  be a list of  $n$  vectors that spans  $U$ . Let  $c$  and  $\theta_B$  as in Theorem 3.6. Then, the polynomial  $R_Z^B(u)$  is given by*

$$R_Z^B(u) = \frac{1}{(n-r)!} \sum_{\substack{B' \in \mathbb{B}(Z^{B+}) \\ \tau \in \text{cone}(B')}} \frac{(-\theta_{B'} u)^{n-r}}{|\det(B')| \prod_{x \in Z \setminus B'} (\theta_{B'} x - c_x)}, \quad (3.12)$$

where  $\tau$  denotes the vector defined in (3.8) and  $Z^{B+}$  denotes the reorientation of the list  $Z$  s. t. all vectors are positive with respect to  $B$ , i. e.  $Z^{B+} = \bigcup_{i=1}^r (P_i, -N_i)$ .

#### 4. A BASIS FOR THE DAHMEN-MICCHELLI SPACE $\mathcal{D}(X)$

In this section we define a set  $B(X)$  and we show that this set is a basis for the central  $\mathcal{D}$ -space  $\mathcal{D}(X)$ . Furthermore, we show that this basis is dual to the basis  $\mathcal{B}(X)$  of the central  $\mathcal{P}$ -space  $\mathcal{P}(X)$ . Note that  $B$  is the equivalent of the letter  $B$  in the Cyrillic alphabet.

**Definition 4.1** (Basis for  $\mathcal{D}(X)$ ). Let  $X \subseteq U \cong \mathbb{R}^r$  be a finite list of vectors that spans  $U$ . Recall that  $\mathbb{B}(X)$  denotes the set of bases that can be selected from  $X$  and that  $E(B)$  denotes the set of externally active elements with respect to a basis  $B$ . We define

$$B(X) := \{\det(B) R_{X \setminus E(B)}^B : B \in \mathbb{B}(X)\}. \quad (4.1)$$

**Theorem 4.2.** *Let  $X \subseteq U \cong \mathbb{R}^r$  be a finite list of vectors that spans  $U$ . Then  $B(X)$  is a basis for the central Dahmen-Micchelli space  $\mathcal{D}(X)$  and this basis is dual to the basis  $\mathcal{B}(X)$  for the central  $\mathcal{P}$ -space  $\mathcal{P}(X)$ .*

*Remark 4.3.*  $\mathcal{D}(X)$  and  $\mathcal{P}(X)$  are independent of the order of the elements of  $X$ . The bases  $\mathcal{B}(X)$  and  $B(X)$  both depend on that order. In Theorem 4.2, we assume that both bases are constructed using the same order.

**Example 4.4.** This is a continuation of Example 1.1. See also Figure 2. The elements of  $B(X)$  are

$$R_{(x_1, x_2)}^{(x_1, x_2)} = 1, \quad (4.2)$$

$$R_X^{(x_1, x_3)} = (T_{x_1} - T_{-x_1}) * (T_{(x_2, x_3)} - T_{(-x_2, -x_3)}) = s_2, \quad (4.3)$$

$$\text{and } R_X^{(x_2, x_3)} = (T_{x_2} - T_{-x_2}) * (T_{(x_1, x_3)} - T_{(-x_1, -x_3)}) = s_1. \quad (4.4)$$

The elements of  $\mathcal{B}(X)$  are

$$Q_{(x_1, x_2)} = p_\emptyset = 1, \quad (4.5)$$

$$Q_{(x_1, x_3)} = p_{x_2} = t_2, \quad (4.6)$$

$$\text{and } Q_{(x_2, x_3)} = p_{x_1} = t_1. \quad (4.7)$$

$\mathcal{B}(X)$  and  $B(X)$  are obviously dual bases.

The proof of Theorem 4.2 is split into four lemmas. Recall that for a basis  $B = (b_1, \dots, b_r)$  we defined a flag of subspaces  $\{0\} = S_0^B \subsetneq S_1^B \subsetneq \dots \subsetneq S_r^B = U \cong \mathbb{R}^r$ , where  $S_i^B := \text{span}(b_1, \dots, b_i)$ .

**Lemma 4.5** (Annihilation criterion). *Let  $Z \subseteq U \cong \mathbb{R}^r$  be a finite list of vectors and let  $B \subseteq Z$  be a basis. Let  $R_Z^B$  be the polynomial that is defined in (3.6). Let  $C \subseteq Z$ . Suppose there exists  $i \in [r]$  s. t.  $Z \cap (S_i^B \setminus S_{i-1}^B) \subseteq C$ .*

*Then,  $D_C R_Z^B = 0$ .*

*Proof.* Note that  $D_a T_{-a} = -\delta_0$ . Using (2.13), we obtain

$$D_C R_i^B = D_C((-1)^{|N_i|} T_{P_i} * T_{-N_i} - (-1)^{|P_i|} T_{-P_i} * T_{N_i}) \quad (4.8)$$

$$= D_{C \setminus (S_i \setminus S_{i-1})}(\delta_0 - \delta_0) = 0. \quad (4.9)$$

This implies

$$D_C R_Z^B = D_{C \setminus (S_i \setminus S_{i-1})} R_1^B * \dots * R_{i-1}^B * 0 * R_{i+1}^B * \dots * R_r^B = 0. \quad (4.10)$$

□

**Lemma 4.6** (Inclusion). *The polynomial  $R_{X \setminus E(B)}^B$  is contained in  $\mathcal{D}(X \setminus E(B))$  for all  $B \in \mathbb{B}(X)$ . Since  $\mathcal{D}(X \setminus E(B)) \subseteq \mathcal{D}(X)$ , this implies*

$$B(X) \subseteq \mathcal{D}(X). \quad (4.11)$$

*Proof.* Let  $\det(B)R_{X \setminus E(B)}^B \in B(X)$  and let  $C \subseteq X \setminus E(B)$  be a cocircuit, i. e.  $C$  intersects all bases that can be selected from  $X \setminus E(B)$ . We need to show that  $D_C R_{X \setminus E(B)}^B = 0$ .  $C$  can be written as  $C = X \setminus (H \cup E(B))$  for some hyperplane  $H \subseteq U$ .

Let  $i$  be minimal s. t.  $S_i \not\subseteq H$ . Such an  $i$  must exist since  $S_r = U$ . Even  $(S_i \setminus S_{i-1}) \cap H = \emptyset$  holds. This implies

$$(X \setminus E(B)) \cap (S_i \setminus S_{i-1}) \subseteq X \setminus (H \cup E(B)) = C. \quad (4.12)$$

By Lemma 4.5, this implies  $D_C R_{X \setminus E(B)}^B = 0$ .  $\square$

The following lemma will be used only in the proof of Lemma 4.8.

**Lemma 4.7.** *Let  $B, D \in \mathbb{B}(X)$ . Suppose that both bases are distinct but have the same number of externally active elements.*

*Then there exists  $i \in [r]$  s. t.*

$$(X \setminus E(D)) \cap (S_i^D \setminus S_{i-1}^D) \subseteq X \setminus (B \cup E(B)). \quad (4.13)$$

*Proof.* Let  $B = (b_1, \dots, b_r)$  and  $D = (d_1, \dots, d_r)$ . Suppose that the lemma is false. Then there exist vectors  $z_1, \dots, z_r$  s. t.

$$z_i \in (X \setminus E(D)) \cap (S_i^D \setminus S_{i-1}^D) \cap (B \cup E(B)). \quad (4.14)$$

Those vectors form a basis because  $z_i \in S_i^D \setminus S_{i-1}^D$ . Since  $z_i$  is not contained in  $E(D)$ ,  $z_i \leq d_i$  must hold. This implies  $E(D) \subseteq E(z_1, \dots, z_r)$ . On the other hand,  $E(z_1, \dots, z_r) \subseteq E(B)$  since all  $z_i$  are contained in  $B \cup E(B)$ .

We have shown that  $E(D) \subseteq E(B)$ . This is a contradiction since no finite set can be contained in a distinct set of the same cardinality.  $\square$

**Lemma 4.8 (Duality).** *Let  $B, D \in \mathbb{B}(X)$ . Let  $Q_B = p_{X \setminus (B \cup E(B))} \in \mathcal{B}(X)$  and let  $R_{X \setminus E(D)}^D$  be the polynomial that is defined in (3.6). Then*

$$\langle Q_B, R_{X \setminus E(D)}^D \rangle = \frac{\delta_{B,D}}{\det(D)}. \quad (4.15)$$

$\delta_{B,D}$  denotes the Kronecker delta and we consider  $B$  and  $D$  to be equal if there exist  $1 \leq i_1 < \dots < i_r \leq N$  s. t.  $B = (x_{i_1}, \dots, x_{i_r}) = D$ .

*Proof.* By Corollary 3.4,  $R_{X \setminus E(D)}^D$  is a homogeneous polynomial of degree  $N - r - |E(D)|$ . Thus, if  $|E(B)| \neq |E(D)|$ , then  $Q_B$  and  $R_{X \setminus E(D)}^D$  are homogeneous polynomials of different degrees and  $\langle Q_B, R_{X \setminus E(D)}^D \rangle = 0$ .

Now suppose that  $B \neq D$  and both bases have the same number of externally active elements. In this case, the statement follows from Lemma 4.5 and Lemma 4.7.

The only case that remains is  $B = D$ . Recall that  $R_{X \setminus E(B)}^B = R_1^B * \dots * R_r^B$ . Consider the  $i$ th factor  $R_i^B$ . The elements of  $(X \setminus E(B)) \cap (S_i \setminus S_{i-1})$  are used for the construction of  $R_i^B$ . Exactly one basis element is contained in this set:  $b_i$ . Recall that in Section 3 we defined a partition  $P_i \cup N_i = (X \setminus E(B)) \cap (S_i \setminus S_{i-1})$ . By construction,  $b_i$  is positive, i. e.  $b_i \in P_i$ . Now we apply the differential operator  $D_{(P_i \setminus b_i) \cup N_i}$  to  $R_i^B$ :

$$D_{(P_i \setminus b_i) \cup N_i}((-1)^{|N_i|} \cdot T_{P_i} * T_{N_i} - (-1)^{|P_i|} \cdot T_{-P_i} * T_{N_i}) = (T_{b_i} + T_{-b_i}). \quad (4.16)$$



Now we can put things together. Note that  $X \setminus (B \cup E(B)) = \bigcup_{i=1}^r ((P_i \setminus b_i) \cup N_i)$ . Hence,

$$D_{X \setminus (B \cup E(B))} R_B = (T_{b_1} + T_{-b_1}) * \cdots * (T_{b_r} + T_{-b_r}) = \frac{1}{\det(B)}. \quad (4.17)$$

This finishes the proof.  $\square$

*Proof of Theorem 4.2.* We know that  $\mathcal{P}(X)$  and  $\mathcal{D}(X)$  are dual via the pairing  $\langle \cdot, \cdot \rangle$  and that  $\mathcal{B}(X)$  is a basis for  $\mathcal{P}(X)$ . By Lemma 4.8,  $B(X)$  and  $\mathcal{B}(X)$  are dual to each other and by Lemma 4.6,  $B(X)$  is contained in  $\mathcal{D}(X)$ . Hence,  $B(X)$  is a basis for  $\mathcal{D}(X)$ .  $\square$

## 5. DELETION-CONTRACTION AND EXACT SEQUENCES

By Proposition 1.13, the Hilbert series of  $\mathcal{D}(X)$  and  $\mathcal{P}(X)$  are equal and an evaluation of the Tutte polynomial. In particular, they satisfy a deletion-contraction identity that extends in a natural way to our algebraic setting. This is reflected by two dual short exact sequences.

In this section we define deletion and contraction and we explain those two exact sequences. While the two sequences were known before, their duality has not yet been stated explicitly in the literature.

Two important matroid operations are deletion and contraction. For realisations of matroids, they are defined as follows. Let  $X \subseteq U$  be a finite list of vectors and let  $x \in X$ . The *deletion* of  $x$  is the list  $X \setminus x$ . The *contraction* of  $x$  is the list  $X/x$ , which is defined to be the image of  $X \setminus x$  under the projection  $\pi_x : U \rightarrow U/x$ . The space  $U$  and other terminology used here are defined in Subsection 1.1.

The space  $\mathcal{P}(X/x)$  is contained in the symmetric algebra  $\text{Sym}(U/x)$ . If  $x = s_r$ , then there is a natural isomorphism  $\text{Sym}(U/x) \cong \mathbb{R}[s_1, \dots, s_{r-1}]$  that maps  $\bar{s}_i$  to  $s_i$ . This isomorphism depends on the choice of the basis  $(s_1, \dots, s_r)$  for  $U$ . Under this identification,  $\text{Sym}(\pi_x)$  is the map from  $\mathbb{R}[s_1, \dots, s_r]$  to  $\mathbb{R}[s_1, \dots, s_{r-1}]$  that sends  $s_r$  to zero and  $s_1, \dots, s_{r-1}$  to themselves.

For  $\mathcal{D}(X/x)$ , the situation is simpler: this space is contained in  $\text{Sym}((U/x)^*) \cong \text{Sym}(x^\circ)$ . This is a subspace of  $\text{Sym}(V)$ . We denote the inclusion map by  $j_x$ . If  $x = s_r$ , then  $\text{Sym}((U/x)^*)$  is isomorphic to  $\mathbb{R}[t_1, \dots, t_{r-1}]$ . This is a canonical isomorphism that is independent of the choice of the basis elements  $s_1, \dots, s_{r-1}$ .

For a graded vector space  $S$ , we write  $S[1]$  for the vector space with the degree shifted up by one.

**Proposition 5.1** ([2]). *Let  $X \subseteq U \cong \mathbb{R}^r$  be a finite list of vectors that spans  $U$  and let  $x \in X$  be neither a loop nor a coloop. Then, the following sequence of graded vector spaces is exact:*

$$0 \rightarrow \mathcal{P}(X \setminus x)[1] \xrightarrow{p_x} \mathcal{P}(X) \xrightarrow{\text{Sym}(\pi_x)} \mathcal{P}(X/x) \rightarrow 0. \quad (5.1)$$

**Proposition 5.2** ([16]). *Let  $X \subseteq U \cong \mathbb{R}^r$  be a finite list of vectors that spans  $U$  and let  $x \in X$  be neither a loop nor a coloop. Then, the following sequence of graded vector spaces is exact:*

$$0 \rightarrow \mathcal{D}(X/x) \xrightarrow{j_x} \mathcal{D}(X) \xrightarrow{D_x} \mathcal{D}(X \setminus x)[1] \rightarrow 0. \quad (5.2)$$

Note that (5.2) is a special case of (1.12) in [16] and it is exact by the results in that paper.

*Remark 5.3.* Proposition 5.1 and Proposition 5.2 are equivalent because of the duality of  $\mathcal{P}(X)$  and  $\mathcal{D}(X)$ .

*Proof of Remark 5.3.* We only show that Proposition 5.2 implies Proposition 5.1. The other implication is similar.

Since dualisation of finite dimensional vector spaces is a contravariant exact functor, the following sequence is exact by Proposition 5.2:

$$0 \rightarrow \mathcal{D}(X \setminus x)^* \xrightarrow{(D_x)^*} \mathcal{D}(X)^* \xrightarrow{(j_x)^*} \mathcal{D}(X/x)^* \rightarrow 0. \quad (5.3)$$

By Proposition 1.12,  $\mathcal{P}(X)$  is isomorphic to  $\mathcal{D}(X)^*$  via  $q \mapsto \langle q, \cdot \rangle$ . Hence, it is sufficient to show that the following two diagrams commute:

$$\begin{array}{ccc} \mathcal{P}(X \setminus x) & \xrightarrow{q \mapsto \langle q, \cdot \rangle} & \mathcal{D}(X \setminus x)^* \\ \downarrow \cdot p_x & & \downarrow (D_x)^* \\ \mathcal{P}(X) & \xrightarrow{q \mapsto \langle q, \cdot \rangle} & \mathcal{D}(X)^* \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{P}(X) & \xrightarrow{q \mapsto \langle q, \cdot \rangle} & \mathcal{D}(X)^* \\ \downarrow \text{Sym}(\pi_x) & & \downarrow (j_x)^* \\ \mathcal{P}(X/x) & \xrightarrow{q \mapsto \langle q, \cdot \rangle} & \mathcal{D}(X/x)^* \end{array}. \quad (5.4)$$

For the diagram on the left, we have to show that  $\langle p_x q, \cdot \rangle = \langle q, D_x \cdot \rangle$  for all  $q \in \mathcal{P}(X \setminus x)$ . This is easy.

For the diagram on the right, we have to show that  $\langle \text{Sym}(\pi_x)q, \cdot \rangle = \langle q, j_x(\cdot) \rangle$  for all  $q \in \mathcal{P}(X)$ . If we choose a basis with  $s_r = x$  this follows from the fact that  $\frac{\partial}{\partial t_r} f = 0$  for all  $f \in \mathbb{R}[t_1, \dots, t_{r-1}]$ .  $\square$

## 6. FORWARD EXCHANGE MATROIDS

In this section we introduce forward exchange matroids. A forward exchange matroid is an ordered matroid together with a subset of its set of bases that satisfies a weak version of the basis exchange axiom (1.1).

The motivation for this definition is the following: the author noticed that most of the results in Subsection 1.4 and Section 4 hold in a far more general context. An important ingredient of the definitions of the spaces  $\mathcal{P}(X)$  and  $\mathcal{D}(X)$  and their bases is the set of bases  $\mathbb{B}(X)$  of the list  $X$ . These two spaces still have nice properties if we modify their definitions and use only a suitable subset  $\mathbb{B}'$  of  $\mathbb{B}(X)$ . It turned out that forward exchange matroids are the right axiomatisation of “suitable subset”.

Let  $(A, \mathbb{B})$  be an ordered matroid of rank  $r$  and let  $B = (b_1, \dots, b_r) \in \mathbb{B}$  be an ordered basis. The flag (3.1) can be defined in combinatorial terms: for  $i \in \{0, \dots, r\}$ , we define  $S_i = S_i^B := \text{cl}\{b_1, \dots, b_i\} \subseteq A$ . Hence, we obtain a flag of flats

$$\{x \in A : x \text{ loop}\} = S_0^B \subsetneq S_1^B \subsetneq S_2^B \subsetneq \dots \subsetneq S_r^B = A. \quad (6.1)$$

One can easily show that for a basis  $B \in \mathbb{B}$  and  $i \in [r]$ , the following statement holds:

$$\text{let } x \in S_i^B \setminus S_{i-1}^B. \text{ Then } B' = (B \setminus b_i) \cup x \text{ is also in } \mathbb{B}. \quad (6.2)$$

Note that  $x \in S_i^B \setminus S_{i-1}^B$  satisfies  $x > b_i$  if and only if  $x$  is externally active with respect to  $B$ . This motivates the name of the following definition.

**Definition 6.1** (Forward exchange property). Let  $(A, \mathbb{B})$  be a matroid and let  $\mathbb{B}' \subseteq \mathbb{B}$ . We say that the set of bases  $\mathbb{B}'$  has the *forward exchange property* if the following holds for all bases  $B \in \mathbb{B}'$  and all  $i \in [r]$ :

$$\text{let } x \in S_i^B \setminus (S_{i-1}^B \cup E(B)). \text{ Then } B' = (B \setminus b_i) \cup x \text{ is also in } \mathbb{B}'. \quad (6.3)$$

*Remark 6.2.* Note that  $S_j^B = S_j^{B'}$  holds for all  $j \geq i$ . If  $x$  is the  $i$ th vector in  $B'$ , this equality holds for all  $j$ , *i. e.*  $B$  and  $B'$  define the same flag.

**Definition 6.3** (Forward exchange matroid). A triple  $(A, \mathbb{B}, \mathbb{B}')$  is called a *forward exchange matroid* if  $(A, \mathbb{B})$  is a matroid and  $\mathbb{B}'$  is a subset of the set of bases  $\mathbb{B}$  with the forward exchange property.

*Remark 6.4.* In this paper, we mainly consider realisations of forward exchange matroids, *i. e.* pairs  $(X, \mathbb{B}')$  where  $X$  is a list of vectors and  $\mathbb{B}' \subseteq \mathbb{B}(X)$  is a set of bases with the forward exchange property.

**Definition 6.5** (Tutte polynomial for forward exchange matroids). Let  $(A, \mathbb{B}, \mathbb{B}')$  be a forward exchange matroid.

Then, we define its *Tutte polynomial* to be

$$T_{(A, \mathbb{B}, \mathbb{B}')} (x, y) := \sum_{B \in \mathbb{B}'} x^{|I(B)|} y^{|E(B)|}, \quad (6.4)$$

where  $I(B)$  and  $E(B)$  denote the sets of internally and externally active elements with respect to  $B$  in the matroid  $(A, \mathbb{B})$ .

*Remark 6.6.* It would be interesting to clarify the relationship between forward-exchange matroids and other set systems studied in combinatorics such as greedoids [4, 31].

## 7. GENERALISED $\mathcal{D}$ -SPACES AND $\mathcal{P}$ -SPACES

Earlier, we considered the spaces  $\mathcal{D}(X)$  and  $\mathcal{P}(X)$  for a given list of vectors  $X$ . The construction of those spaces relied mainly on the matroidal properties of the list  $X$ , namely on the sets of bases and cocircuits.

Motivated by questions in approximation theory, various authors generalised those constructions. Given a list  $X$  and a subset  $\mathbb{B}'$  of its set of bases  $\mathbb{B}(X)$ , one can define a set  $\mathcal{D}(X, \mathbb{B}')$  as the kernel of the ideal generated by the  $\mathbb{B}'$ -cocircuits (*i. e.* sets that intersect all bases in  $\mathbb{B}'$ ). Under certain conditions,  $\dim \mathcal{D}(X, \mathbb{B}') = |\mathbb{B}'|$  still holds. In this section we show that if the set  $\mathbb{B}'$  has the forward exchange property, this equality holds and there is a canonical dual space  $\mathcal{P}(X, \mathbb{B}')$ . Both, the generalised  $\mathcal{D}$ -spaces and the generalised  $\mathcal{P}$ -spaces satisfy deletion-contraction identities as in Section 5 and there are canonical bases for both spaces that are dual.

### 7.1. Definitions and Main Result.

**Definition 7.1** (generalised  $\mathcal{D}$ -spaces). Let  $X \subseteq U \cong \mathbb{R}^r$  be a finite list of vectors that spans  $U$  and let  $\mathbb{B}'$  be an arbitrary subset of its set of bases  $\mathbb{B}(X)$ . A set  $C \subseteq X$  is called a  $\mathbb{B}'$ -cocircuit if  $C$  intersects every basis in  $\mathbb{B}'$  and  $C$  is inclusion-minimal with this property.

The *generalised  $\mathcal{D}$ -space* defined by  $X$  and  $\mathbb{B}'$  is

$$\mathcal{D}(X, \mathbb{B}') := \{f : D_C f = 0 \text{ for all } \mathbb{B}'\text{-cocircuits } C\} = \ker \mathcal{J}(X, \mathbb{B}'), \quad (7.1)$$

$$\text{where } \mathcal{J}(X, \mathbb{B}') := \text{ideal}\{p_C : C \subseteq X \text{ is a } \mathbb{B}'\text{-cocircuit}\}. \quad (7.2)$$

**Proposition 7.2** ([14, Theorem 6.6]). Let  $X \subseteq U \cong \mathbb{R}^r$  be a finite list of vectors that spans  $U$  and let  $\mathbb{B}'$  be an arbitrary subset of its set of bases  $\mathbb{B}(X)$ . Then

$$\dim \mathcal{D}(X, \mathbb{B}') \geq |\mathbb{B}'|. \quad (7.3)$$

**Definition 7.3** (generalised  $\mathcal{P}$ -spaces). Let  $X \subseteq U \cong \mathbb{R}^r$  be a finite list of vectors that spans  $U$  and let  $\mathbb{B}'$  be an arbitrary subset of its set of bases  $\mathbb{B}(X)$ . Then we define

$$\mathcal{B}(X, \mathbb{B}') := \{Q_B : B \in \mathbb{B}'\} = \{p_{X \setminus (B \cup E(B))} : B \in \mathbb{B}'\} \quad (7.4)$$

$$\text{and } \mathcal{P}(X, \mathbb{B}') := \text{span } \mathcal{B}(X, \mathbb{B}'). \quad (7.5)$$

We call  $\mathcal{P}(X, \mathbb{B}')$  the *generalised  $\mathcal{P}$ -space* defined by  $X$  and  $\mathbb{B}'$ .

*Remark 7.4.* The set  $\mathcal{B}(X, \mathbb{B}')$  is a basis for  $\mathcal{P}(X, \mathbb{B}')$ . By definition, it is spanning and it is linearly independent because it is a subset of  $\mathcal{B}(X)$ .

If the set  $\mathbb{B}'$  has the forward exchange property, the spaces  $\mathcal{D}(X, \mathbb{B}')$  and  $\mathcal{P}(X, \mathbb{B}')$  have many nice properties. Here is the Main Theorem of this section.

**Theorem 7.5.** *Let  $X \subseteq U \cong \mathbb{R}^r$  be a finite list of vectors that spans  $U$  and let  $\mathbb{B}' \subseteq \mathbb{B}(X)$  be a set of bases with the forward exchange property.*

*Then, the generalised  $\mathcal{D}$ -space  $\mathcal{D}(X, \mathbb{B}')$  and the generalised  $\mathcal{P}$ -space  $\mathcal{P}(X, \mathbb{B}')$  are dual via the pairing  $\langle \cdot, \cdot \rangle$ . In addition,*

$$\mathcal{B}(X, \mathbb{B}') := \{\det(B) R_{X \setminus E(B)}^B : B \in \mathbb{B}'\} \quad (7.6)$$

*is a basis for  $\mathcal{D}(X, \mathbb{B}')$  and this basis is dual to the basis  $\mathcal{B}(X, \mathbb{B}')$  for  $\mathcal{P}(X, \mathbb{B}')$ .*

**Corollary 7.6.** *Let  $X \subseteq U \cong \mathbb{R}^r$  be a list of  $N$  vectors that spans  $U$  and let  $\mathbb{B}' \subseteq \mathbb{B}(X)$  be a set of bases with the forward exchange property. Then*

$$\text{Hilb}(\mathcal{P}(X, \mathbb{B}'), q) = \text{Hilb}(\mathcal{D}(X, \mathbb{B}'), q) = \sum_{B \in \mathbb{B}'} q^{N-r-|E(B)|} = q^{N-r} T_{(X, \mathbb{B}(X), \mathbb{B}')} (1, \frac{1}{q}).$$

Here are two examples that help to understand generalised  $\mathcal{D}$ -spaces, generalised  $\mathcal{P}$ -spaces, and Theorem 7.5.

**Example 7.7.** Let  $X = (e_1, e_2, e_3, a, b) \subseteq \mathbb{R}^3$  where  $e_1, e_2, e_3$  denote the unit vectors and  $a = (\alpha, \beta, \gamma)$  and  $b$  are generic. In particular,  $\alpha, \beta, \gamma \neq 0$ . Let

$$\mathbb{B}' := \{(e_1 e_2 e_3), (e_1 e_2 a), (e_1 e_2 b), (e_1 e_3 a), (e_1 e_3 b), (e_2 e_3 a), (e_2 e_3 b)\} \subseteq \mathbb{B}(X).$$

The reader is invited to check that  $\mathbb{B}'$  has the forward exchange property. The Tutte polynomial is  $T_{(X, \mathbb{B}(X), \mathbb{B}')} (x, y) = 3x + 3y + y^2$  and the  $\mathbb{B}'$ -cocircuits are  $\{e_1 e_2, e_1 e_3, e_2 e_3, e_1 a b, e_2 a b, e_3 a b\}$ . Hence,

$$\begin{aligned} \mathcal{D}(X, \mathbb{B}') &= \ker \text{ideal}\{s_1 s_2, s_1 s_3, s_2 s_3, s_1 p_{ab}, s_2 p_{ab}, s_3 p_{ab}\} \\ &= \text{span}\{1, t_1, t_2, t_3, t_1^2, t_2^2, t_3^2\}, \\ \mathcal{B}(X, \mathbb{B}') &= \{1, p_{e_3}, p_{e_3 a}, p_{e_2}, p_{e_2 a}, p_{e_1}, p_{e_1 a}\}, \\ \mathcal{P}(X, \mathbb{B}') &= \text{span}\{1, s_1, s_2, s_3, s_1(\alpha s_1 + \beta s_2 + \gamma s_3), s_2(\alpha s_1 + \beta s_2 + \gamma s_3), \\ &\quad s_3(\alpha s_1 + \beta s_2 + \gamma s_3)\}, \text{ and} \end{aligned}$$

$$\mathcal{B}(X, \mathbb{B}') = \left\{1, t_3, \frac{t_3^2}{2\gamma}, t_2, \frac{t_2^2}{2\beta}, t_1, \frac{t_1^2}{2\alpha}\right\}.$$

**Example 7.8.** Let  $N \geq 3$  be an integer. Let  $X_N = (x_1, \dots, x_N)$  be a list of vectors in general position in  $\mathbb{R}^2$  with  $x_1 = e_1$ ,  $x_2 = e_2$ , and  $x_3 = e_1 + e_2$ . In addition, we suppose that the second coordinate of all vectors  $x_i$  ( $i \geq 3$ ) is one. Let  $\mathbb{B}' := \{(x_1, x_i) : i \in \{2, \dots, N\}\} \cup \{(x_2, x_3)\}$ . Note that  $\mathbb{B}'$  is totally unimodular, *i. e.*

all elements have determinant 1 or  $-1$  and  $\mathbb{B}'$  has the forward exchange property. Then,

$$\begin{aligned} \mathcal{D}(X_N, \mathbb{B}') &= \ker \text{ideal}\{p_{x_1 x_2}, p_{x_1 x_3}, p_{x_2 \dots x_N}\} = \text{span}\{1, t_1, t_2, t_2^2, t_2^3, \dots, t_2^{N-2}\}, \\ B(X_N, \mathbb{B}') &= \left\{1, t_1, t_2, \frac{t_2}{2}, \dots, \frac{t_2^{N-2}}{(N-2)!}\right\}, \text{ and} \end{aligned} \quad (7.7)$$

$$\mathcal{B}(X_N, \mathbb{B}') = \{1, p_{x_1}\} \cup \{p_{x_2 \dots x_i} : i \in \{2, \dots, N-1\}\}. \quad (7.8)$$

Now we embark on the proof of Theorem 7.5. We start with the following simple lemma.

**Lemma 7.9** (Inclusion). *Let  $X \subseteq U \cong \mathbb{R}^r$  be a finite list of vectors that spans  $U$  and let  $\mathbb{B}' \subseteq \mathbb{B}(X)$  be a set of bases with the forward exchange property. Then*

$$B(X, \mathbb{B}') \subseteq \mathcal{D}(X, \mathbb{B}'). \quad (7.9)$$

*Proof.* Let  $B \in \mathbb{B}'$  and let  $\det(B)R_{X \setminus E(B)}^B \in B(X, \mathbb{B}')$  be the corresponding basis element. Let  $C \subseteq X$  be a  $\mathbb{B}'$ -cocircuit, *i. e.* an inclusion-minimal subset of  $X$  that intersects every basis in  $\mathbb{B}'$ .

Let  $B = (b_1, \dots, b_r)$ . If there exists an  $i$  s. t.  $(S_i^B \setminus S_{i-1}^B) \cap (X \setminus E(B)) \subseteq C$ , we are done by Lemma 4.5. Now suppose that this is not the case, *i. e.* for every  $i \in [r]$ , there is a  $z_i \in (S_i^B \setminus S_{i-1}^B) \cap (X \setminus (E(B) \cup C))$ . Then we define a sequence of bases  $B_0, \dots, B_r$  by

$$B_0 := B \quad \text{and} \quad B_i := (B_{i-1} \setminus b_i) \cup z_i \text{ for } i \in [r]. \quad (7.10)$$

The lists  $B_i$  are indeed bases and even though in general, they might define different flags, they satisfy

$$(S_i^{B_{i-1}} \setminus S_{i-1}^{B_{i-1}}) \cap (X \setminus E(B_{i-1})) = (S_i^B \setminus S_{i-1}^B) \cap (X \setminus E(B)), \quad (7.11)$$

because  $\text{span}(b_1, \dots, b_i) = \text{span}(z_1, \dots, z_i)$  for all  $i \in [r]$ . Hence,  $B_i \in \mathbb{B}'$  implies  $B_{i+1} \in \mathbb{B}'$  because  $\mathbb{B}'$  has the forward exchange property. In particular,  $B_r = (z_1, \dots, z_r) \in \mathbb{B}'$ . By construction,  $B_r \cap C = \emptyset$ . This is a contradiction.  $\square$

**Definition 7.10.** Let  $(A, \mathbb{B})$  be a matroid and let  $\mathbb{B}' \subseteq \mathbb{B}$ . Let  $x \in A$ .  $\mathbb{B}'$  can be partitioned as  $\mathbb{B}' = \mathbb{B}'_{\setminus x} \cup \mathbb{B}'_{|x}$ , where

$$\mathbb{B}'_{\setminus x} := \{B \in \mathbb{B}' : x \notin B\} \text{ denotes the } \textit{deletion} \text{ of } x \text{ and} \quad (7.12)$$

$$\mathbb{B}'_{|x} := \{B \in \mathbb{B}' : x \in B\} \text{ the } \textit{restriction} \text{ to } x. \quad (7.13)$$

If we are given a list of vectors  $X \subseteq U$  and a set of bases  $\mathbb{B}' \subseteq \mathbb{B}(X)$ , we can also define the *contraction*  $\mathbb{B}'_{/x}$ . Recall that  $\pi_x : U \rightarrow U/x$  denotes the canonical projection. Then, we define

$$\mathbb{B}'_{/x} := \{\pi_x(B \setminus x) : x \in B \in \mathbb{B}'_{|x}\}. \quad (7.14)$$

*Remark 7.11.* For technical reasons, it is helpful to distinguish the contraction  $\mathbb{B}'_{/x}$  and the restriction  $\mathbb{B}'_{|x}$  although there is a canonical bijection between both sets.

We now introduce the concept of placibility. This is a condition on a set of bases  $\mathbb{B}'$  which implies equality in (7.3).

**Definition 7.12** ([16], see also [33]). Let  $(A, \mathbb{B})$  be a matroid and let  $\mathbb{B}' \subseteq \mathbb{B}$  be a non-empty set of bases.

- (i) We call an element  $x \in A$  *placeable* in  $\mathbb{B}'$  if for each  $B \in \mathbb{B}'$ , there exists an element  $b \in B$  such that  $(B \setminus b) \cup x \in \mathbb{B}'$ .
- (ii) We say that  $\mathbb{B}'$  is *placible* if one of the following two conditions holds:
  - (a)  $\mathbb{B}'$  is a singleton or
  - (b) there exists  $x \in A$  s. t.  $x$  is placeable in  $\mathbb{B}'$  and both,  $\mathbb{B}'_{|x}$  and  $\mathbb{B}'_{\setminus x}$  are non-empty and placible.

**Proposition 7.13** ([16]). *Let  $X \subseteq U \cong \mathbb{R}^r$  be a finite list of vectors that spans  $U$  and let  $\mathbb{B}'$  be an arbitrary subset of its set of bases  $\mathbb{B}(X)$ . If  $\mathbb{B}'$  is placible, then  $\dim \mathcal{D}(X, \mathbb{B}') = |\mathbb{B}'|$ .*

**Lemma 7.14.** *Let  $(A, \mathbb{B}, \mathbb{B}')$  be a forward exchange matroid. Then  $\mathbb{B}'$  is placible.*

*Proof.* If  $|\mathbb{B}'| = 1$ , then  $\mathbb{B}'$  is placible by definition. Now let  $|\mathbb{B}'| \geq 2$ . Let  $x$  be the minimal element in  $A$  s. t. both,  $\mathbb{B}'_{|x}$  and  $\mathbb{B}'_{\setminus x}$  are non-empty. Such an element must exist if  $|\mathbb{B}'| \geq 2$ .

We show now that  $x$  is placeable in  $\mathbb{B}'$ . Let  $B = (b_1, \dots, b_r)$  be a basis in  $\mathbb{B}'$  and let  $i \in [r]$  s. t.  $x \in S_i^B \setminus S_{i-1}^B$ . We claim that  $x \leq b_i$ . Suppose not. Because of the minimality of  $x$ , this implies that  $b_1, \dots, b_i$  are contained in all bases in  $\mathbb{B}'$ . Since  $x \in \text{span}(b_1, \dots, b_i)$ , this implies that  $x$  is not contained in any basis. This is a contradiction because we assumed that  $\mathbb{B}'_{|x}$  is non-empty.

Now we have established that  $x \leq b_i$ . This implies that  $x$  is not externally active. Hence, because of the forward exchange property,  $(B \setminus b_i) \cup x \in \mathbb{B}'$ , i. e.  $x$  is placeable in  $B \in \mathbb{B}'$ .

It remains to be shown that  $\mathbb{B}'_{\setminus x}$  and  $\mathbb{B}'_{/x}$  are both placible. By induction, it is sufficient to show that both sets have the forward exchange property. For  $\mathbb{B}'_{\setminus x}$ , this is clear. For  $\mathbb{B}'_{/x}$ , this follows from the following fact: by the choice of  $x$ , all  $a \in A$  that satisfy  $a < x$  are either contained in all bases or in no basis in  $\mathbb{B}'_{/x}$ .  $\square$

*Proof of Theorem 7.5.* By Lemma 7.9,  $B(X, \mathbb{B}') \subseteq \mathcal{D}(X, \mathbb{B}')$ . By Proposition 7.13 and by Lemma 7.14,  $\dim \mathcal{D}(X, \mathbb{B}') = |\mathbb{B}'| = |B(X, \mathbb{B}')|$ . Linear independence of  $B(X, \mathbb{B}')$  is clear because it is a subset of  $B(X)$ . For the same reason, the duality with  $\mathcal{B}(X, \mathbb{B}') \subseteq \mathcal{B}(X)$  follows from Lemma 4.8.  $\square$

**Remark 7.15.** The correspondence between  $\mathcal{D}(X)$  and the set of vertices  $S$  of a hyperplane arrangement  $\mathcal{H}(X, c)$  in general position that is stated in Theorem 2.8 generalises in a straightforward way to a correspondence between  $\mathcal{D}(X, \mathbb{B}')$  and the subset of  $S$  that is defined by  $\mathbb{B}'$ .

**7.2. Deletion-contraction and exact sequences.** In this subsection, we show that the results in Section 5 about deletion-contraction and exact sequences naturally extend to generalised  $\mathcal{D}$ -spaces and  $\mathcal{P}$ -spaces. We use the same terminology as in that section.

Recall that for a graded vector space  $S$ , we write  $S[1]$  for the vector space with the degree shifted up by one.

**Proposition 7.16.** *Let  $X \subseteq U \cong \mathbb{R}^r$  be a finite list of vectors that spans  $U$  and let  $\mathbb{B}' \subseteq \mathbb{B}(X)$  be a set of bases with the forward exchange property. Let  $x$  be the minimal element of  $X$  that is neither a loop nor a coloop. Then, the following sequence of graded vector spaces is exact:*

$$0 \rightarrow \mathcal{D}(X/x, \mathbb{B}'_{/x}) \xrightarrow{j_x} \mathcal{D}(X, \mathbb{B}') \xrightarrow{D_x} \mathcal{D}(X \setminus x, \mathbb{B}'_{\setminus x})[1] \rightarrow 0. \quad (7.15)$$

*Proof.* Let  $B \in \mathbb{B}'$  be a basis that does not contain  $x$ . Because of the minimality,  $x$  is not externally active with respect to  $B$ . This implies  $D_x R_{X \setminus E(B)}^B = R_{X \setminus (E(B) \cup x)}^B$ . Hence,  $D_x : \{\det(B)R^B \in \mathcal{B}(X, \mathbb{B}') : x \notin B\} \rightarrow \mathcal{B}(X \setminus x, \mathbb{B}'_{\setminus x})$  is a bijection and consequently,  $D_x$  maps  $\mathcal{D}(X, \mathbb{B}')$  surjectively to  $\mathcal{D}(X \setminus x, \mathbb{B}'_{\setminus x})$ . For the rest of the proof, we refer the reader to [16], in particular to the explanations following (1.12) and to Theorem 2.16.  $\square$

**Proposition 7.17.** *Let  $X \subseteq U \cong \mathbb{R}^r$  be a finite list of vectors that spans  $U$  and let  $\mathbb{B}' \subseteq \mathbb{B}(X)$  be a set of bases with the forward exchange property. Let  $x$  be the minimal element of  $X$  that is neither a loop nor a coloop. Then, the following sequence of graded vector spaces is exact:*

$$0 \rightarrow \mathcal{P}(X \setminus x, \mathbb{B}'_{\setminus x})[1] \xrightarrow{\cdot p_x} \mathcal{P}(X, \mathbb{B}') \xrightarrow{\text{Sym}(\pi_x)} \mathcal{P}(X/x, \mathbb{B}'_{/x}) \rightarrow 0. \quad (7.16)$$

*Proof.* One can easily check this for the bases of the  $\mathcal{P}$ -spaces. Alternatively, it can be deduced from Proposition 7.16 using a duality argument as in the proof of Remark 5.3.  $\square$

*Remark 7.18.* The exact sequences in this section require  $x$  to be minimal in contrast to the ones Section 5, where  $x$  can be any element that is neither a loop nor a coloop. This reflects the fact that matroids have an (unordered) ground set, while forward exchange matroids have an (ordered) ground list.

*Remark 7.19.* One could replace  $\mathcal{D}(X/x, \mathbb{B}'_{/x})$  by  $\mathcal{D}(X, \mathbb{B}'_{|x})$  in Proposition 7.16. The analogous replacement in Proposition 7.17 would be problematic. The reasons for that are explained in Section 5.

**7.3.  $\mathcal{P}(X, \mathbb{B}')$  as the kernel of a power ideal.** By now, we have seen that most of the results about  $\mathcal{D}(X)$  and  $\mathcal{P}(X)$  that we stated earlier also hold for the generalised  $\mathcal{D}$ -spaces and  $\mathcal{P}$ -spaces. The only thing that is missing is a power ideal  $\mathcal{I}(X, \mathbb{B}')$  s. t.  $\mathcal{P}(X, \mathbb{B}') = \ker \mathcal{I}(X, \mathbb{B}')$ . Unfortunately, such an ideal does not always exist.

In this section, we describe the natural candidate for this power ideal and we give an example where its kernel is equal to  $\mathcal{P}(X, \mathbb{B}')$  and one where it is not.

**Definition 7.20.** Let  $X \subseteq U \cong \mathbb{R}^r$  be a finite list of vectors that spans  $U$  and let  $\mathbb{B}'$  be an arbitrary subset of its set of bases  $\mathbb{B}(X)$ . Recall that  $V := U^*$ . We define a function  $\kappa : V \rightarrow \mathbb{N}$  by

$$\kappa(\eta) := \max_{B \in \mathbb{B}'} |X \setminus (B \cup E(B) \cup \eta^o)| \quad (7.17)$$

$$\text{and } \mathcal{I}(X, \mathbb{B}') := \text{ideal}\{p_\eta^{\kappa(\eta)+1} : \eta \in V \setminus \{0\}\}. \quad (7.18)$$

**Lemma 7.21.** *Let  $X \subseteq U \cong \mathbb{R}^r$  be a finite list of vectors that spans  $U$  and let  $\mathbb{B}'$  be an arbitrary subset of its set of bases  $\mathbb{B}(X)$ . Then*

$$\mathcal{P}(X, \mathbb{B}') \subseteq \ker \mathcal{I}(X, \mathbb{B}'). \quad (7.19)$$

*Proof.* It is sufficient to show that all elements of the basis  $\mathcal{B}(X, \mathbb{B}')$  are contained in  $\ker \mathcal{I}(X, \mathbb{B}')$ . Let  $B \in \mathbb{B}'$  and let  $\eta \in V \setminus \{0\}$ . Then

$$D_\eta^{\kappa(\eta)} p_{X \setminus (B \cup E(B))} = p_{(X \cap \eta^o) \setminus (B \cup E(B))} D_\eta^{\kappa(\eta)+1} p_{X \setminus (B \cup E(B) \cup \eta^o)} = 0 \quad (7.20)$$

The first equality follows from Leibniz's law. The second follows from the fact that by definition,  $\kappa(\eta) \geq |X \setminus (B \cup E(B) \cup \eta^o)|$ .  $\square$



*Remark 7.22.* If one examines the proof of Lemma 7.21, one immediately sees that  $\mathcal{I}(X, \mathbb{B})$  is the only power ideal for which  $\mathcal{P}(X, \mathbb{B}') = \ker \mathcal{I}(X, \mathbb{B}')$  can possibly hold.

*Remark 7.23.* In some cases,  $\mathcal{P}(X, \mathbb{B}')$  and  $\ker \mathcal{I}(X, \mathbb{B}')$  are equal (see Example 7.25). In other cases however,  $\mathcal{P}(X, \mathbb{B}')$  is not even closed under differentiation (see Example 7.8).

Remark 7.23 naturally leads to the following question.

**Question 7.24.** Is there a simple criterion to decide whether  $\mathcal{P}(X, \mathbb{B}')$  is closed under differentiation or if  $\mathcal{P}(X, \mathbb{B}') = \ker \mathcal{I}(X, \mathbb{B}')$  holds?

**Example 7.25.** This is a continuation of Example 7.7. Recall that we considered the list  $X = (e_1, e_2, e_3, a, b) \subseteq \mathbb{R}^3$  where  $a$  and  $b$  are generic vectors and  $a = (\alpha, \beta, \gamma)$  with  $\alpha, \beta, \gamma \neq 0$ . The set of bases is

$$\mathbb{B}' = \{(e_1 e_2 e_3), (e_1 e_2 a), (e_1 e_2 b), (e_1 e_3 a), (e_1 e_3 b), (e_2 e_3 a), (e_2 e_3 b)\} \subseteq \mathbb{B}(X).$$

In order to calculate the function  $\kappa$ , we first determine the inclusion-maximal lists in  $\{X \setminus (B \cup E(B)) : B \in \mathbb{B}\}$ . Those are  $(e_1 a)$ ,  $(e_2 a)$ , and  $(e_3 a)$ . We can deduce that  $\kappa(\eta)$  is one if  $\eta \in a^\circ$  and two otherwise. We obtain

$$\begin{aligned} \mathcal{I}(X, \mathbb{B}') &= \text{ideal}\{p_{(\alpha, -\beta, 0)}^2, p_{(0, \beta, -\gamma)}^2, p_{(\alpha, 0, -\gamma)}^2\} + \mathbb{R}[s_1, s_2, s_3]_{\geq 3} \text{ and} \\ \mathcal{P}(X, \mathbb{B}') &= \ker \mathcal{I}(X, \mathbb{B}') = \text{span}\{1, s_1, s_2, s_3, s_1(\alpha s_1 + \beta s_2 + \gamma s_3), \\ &\quad s_2(\alpha s_1 + \beta s_2 + \gamma s_3), s_3(\alpha s_1 + \beta s_2 + \gamma s_3)\}. \end{aligned}$$

The degree two component of  $\mathbb{R}[s_1, s_2, s_3]/\mathcal{I}(X, \mathbb{B}')$  is three-dimensional. This implies that  $\mathcal{P}(X, \mathbb{B}') = \ker \mathcal{I}(X, \mathbb{B}')$ .

## 8. COMPARISON WITH PREVIOUSLY KNOWN ZONOTOPAL SPACES

In this section we review the definitions of various zonotopal spaces that have been studied previously by other authors. It turns out that they are all special cases of the generalised  $\mathcal{D}$ -spaces and  $\mathcal{P}$ -spaces that we introduced in Section 7. The most prominent examples are of course the central spaces  $\mathcal{D}(X)$  and  $\mathcal{P}(X)$  that we obtain if we choose  $\mathbb{B}' = \mathbb{B}(X)$ .

Let  $A = (a_1, \dots, a_n) \subseteq \mathbb{R}^r$  be a list of vectors that spans  $\mathbb{R}^r$  and let  $X = (x_1, \dots, x_N) \subseteq \mathbb{R}^r$ , where  $N \geq n$  and  $a_i = x_i$  for  $i \in [n]$ . In [28, 29, 33], the spaces  $\mathcal{D}(X, \mathbb{B}')$  and  $\mathcal{P}(X, \mathbb{B}')$  are studied for certain sets of bases  $\mathbb{B}' \subseteq \mathbb{B}(X)$ . Here are the definitions of those sets of bases.

**Definition 8.1** (Internal and external bases [28]). Let  $A \subseteq \mathbb{R}^r$  be a list of vectors that spans  $\mathbb{R}^r$  and let  $B_0 = (b_1, \dots, b_r) \subseteq \mathbb{R}^r$  be an arbitrary basis for  $\mathbb{R}^r$  that is not necessarily contained in  $\mathbb{B}(A)$ . Let  $X = (A, B_0)$  and let

$$\text{ex} : \{I \subseteq A : I \text{ linearly independent}\} \rightarrow \mathbb{B}(X) \quad (8.1)$$

be the function that maps an independent set in  $A$  to its greedy extension. This means that given an independent set  $I \subseteq A$ , the vectors  $b_1, \dots, b_r$  are added successively to  $I$  unless the resulting set would be linearly dependent.

Then we define the set of *external bases*  $\mathbb{B}_+(A, B_0)$  and the set of *internal bases*  $\mathbb{B}_-(A)$  by

$$\mathbb{B}_+(A, B_0) := \{B \in \mathbb{B}(X) : B = \text{ex}(I) \text{ for some } I \subseteq A \text{ independent}\} \quad (8.2)$$

$$\text{and } \mathbb{B}_-(A) := \{B \in \mathbb{B}(A) : B \text{ contains no internally active elements}\}. \quad (8.3)$$

The *lattice of flats*  $\mathcal{L}(A)$  of the matroid  $(A, \mathbb{B}(A))$  is the set  $\{C \subseteq A : \text{cl}(C) = C\}$  ordered by inclusion. An *upper set*  $J \subseteq \mathcal{L}(A)$  is an upward closed set, *i. e.*  $C_1 \subseteq C_2 \in J$  implies  $C_1 \in J$ .

**Definition 8.2** (Semi-internal and semi-external bases [29]). We use the same terminology as in Definition 8.1. In addition, we fix an upper set  $J$  in the lattice of flats  $\mathcal{L}(A)$  of the matroid  $(A, \mathbb{B}(A))$ . For the semi-internal space, we fix an independent set  $I_0 \subseteq A$  whose elements are maximal in  $A$ .

Then we define the set of *semi-external bases*  $\mathbb{B}_+(A, B_0, J)$  and the set of *semi-internal bases*  $\mathbb{B}_-(A, I_0)$  by

$$\mathbb{B}_+(A, B_0, J) := \{B \in \mathbb{B}(X) : B = \text{ex}(I) \text{ for some } I \subseteq A \text{ independent and } \text{cl}(I) \in J\} \quad (8.4)$$

$$\mathbb{B}_-(A, I_0) := \{B \in \mathbb{B}(A) : B \cap I_0 \text{ contains no internally active elements}\}.$$

**Definition 8.3** (Generalised external bases [33]). Let  $A = (a_1, \dots, a_n) \subseteq \mathbb{R}^r$  be a list of vectors. Let  $\kappa : \mathcal{L}(A) \rightarrow \{0, 1, 2, \dots\}$  be an increasing function, *i. e.*  $C_1 \subseteq C_2$  implies  $\kappa(C_1) \leq \kappa(C_2)$ .

Let  $X = (A, Y)$ , where  $Y = (y_1, y_2, \dots, y_{\kappa(A)+r})$  is a list of generic vectors, *i. e.* if  $y_i$  is in the span of  $Z \subseteq X \setminus y_i$ , then  $\text{span}(Z) = \text{span}(X)$ .

Then we define

$$\mathbb{B}_\kappa(A, Y) := \{B \in \mathbb{B}(X) : B \cap Y \subseteq (y_1, \dots, y_{\kappa(\text{cl}(A \cap B)) + |B \cap Y|})\}. \quad (8.5)$$

*Remark 8.4.* The spaces  $\mathcal{P}(X, \mathbb{B}')$  and  $\mathcal{D}(X, \mathbb{B}')$  are equal to

- the external spaces  $\mathcal{P}_+(X)$  and  $\mathcal{D}_+(X)$  in [28] if  $\mathbb{B}'$  is the set of external bases;
- the semi-external spaces  $\mathcal{P}_+(X, J)$  and  $\mathcal{D}_+(X, J)$  in [29] if  $\mathbb{B}'$  is the set of semi-external bases;
- the generalised external spaces  $\mathcal{P}_\kappa(X)$  and  $\mathcal{D}_\kappa(X)$  in [33] if  $\mathbb{B}'$  is the set of generalised external bases. For  $\mathcal{P}_\kappa(X)$ , we need to assume in addition that  $\kappa$  is incremental, *i. e.* for two flats  $C_1 \subseteq C_2$ ,  $\kappa(C_2) - \kappa(C_1) \leq \dim(C_2) - \dim(C_1)$ .

Furthermore, the space  $\mathcal{D}(X, \mathbb{B}')$  is equal to the (semi-)internal space  $\mathcal{D}_-(X)$  resp.  $\mathcal{D}_-(X, I_0)$  in [28, 29] if  $\mathbb{B}'$  is the set of (semi-)internal bases.

*Remark 8.5.* The (semi-)internal spaces  $\mathcal{P}(X, \mathbb{B}_-(X))$  and  $\mathcal{P}(X, \mathbb{B}_-(X, I_0))$  are in general different from the spaces  $\mathcal{P}_-(X)$  and  $\mathcal{P}_-(X, I_0)$  in [28, 29], but they have the same Hilbert series.

*Remark 8.6.* The theorems about duality of certain  $\mathcal{P}$ -spaces and  $\mathcal{D}$ -spaces in [28, 29, 33] are all special cases of Theorem 7.5. This is a consequence of Lemma 8.7 below.

**Lemma 8.7.** *The sets of bases defined in Definitions 8.1, 8.2, and 8.3 all have the forward exchange property.*

*Proof.* We use the following notation throughout the proof:  $B = (b_1, \dots, b_r)$  is a basis and  $x \in (S_i^B \setminus S_{i-1}^B) \cap (X \setminus E(B))$  for some  $i \in [r]$ . In addition,  $B' := (B \cup x) \setminus b_i$ . Since  $x$  is not externally active,  $x \leq b_i$  holds. We may even assume  $x < b_i$  because if equality occurs, nothing needs to be shown.

Internal and external bases are special cases of semi-internal and semi-external bases so we do not consider them separately.

We start with the semi-external bases. Let  $B \in \mathbb{B}_+(A, B_0, J)$ , i. e.  $B$  is the greedy extension of an independent set  $I \subseteq A$ . Recall that  $x < b_i$ . Hence,  $x \in A$  because if  $x$  was in  $B_0$ , the greedy extension of  $I$  would contain  $x$  instead of  $b_i$ . Now one can easily check that  $B' \cap A$  is independent and that  $\text{ex}(B' \cap A) = B'$ . This is equivalent to  $B' \in \mathbb{B}_+(A, B_0, J)$ .

Now we consider the semi-internal bases. Let  $B \in \mathbb{B}_-(A, I_0)$ . By construction, the fundamental cocircuits of  $B$  and  $b_i$  resp.  $x$  are equal. As  $x < b_i$ , inactivity of  $b_i$  implies inactivity of  $x$ . Hence,  $B' \in \mathbb{B}_-(A, I_0)$ .

Last, we consider the generalised external bases. Let  $B \in \mathbb{B}_\kappa(A, Y)$ . If  $b_i \in A$ , then  $\kappa(\text{cl}(A \cap B)) + |B \cap Y| = \kappa(\text{cl}(A \cap B')) + |B' \cap Y|$ . This implies  $B' \in \mathbb{B}_\kappa(A, Y)$ . If  $b_j \in Y$ , then  $j = r$  must hold because the vectors in  $Y$  are generic and we are supposing  $b_j \neq x$ . Replacing  $b_r$  by  $x$  reduces the index of the maximal element in  $Y$  that is permitted in  $B'$  by at most one since  $\kappa$  is non-decreasing on  $\mathcal{L}(A)$ . Since we remove the maximal element of the basis  $B$ , this causes no problems.  $\square$

*Remark 8.8.* Various  $\mathcal{P}$ -spaces without a dual  $\mathcal{D}$ -space have been studied by several other authors e. g. [2, 3, 32, 36, 42]. The approach in [2, 32] is slightly different from ours. The authors of these two papers start with a list of vectors  $X$  and an integer  $k$ . In our construction, it is eventually necessary to add additional elements to the list  $X$  in order to obtain arbitrarily large  $\mathcal{P}$ -spaces. In their construction, it is sufficient to let the integer parameter  $k$  grow while keeping the list  $X$  fixed.

Their construction of a basis for the  $\mathcal{P}$ -space takes into account the internal activity of the bases in  $\mathbb{B}(X)$ . Every element of  $\mathbb{B}(X)$  with internally active elements may define multiple elements of the basis for the  $\mathcal{P}$ -space.

**Example 8.9.** Example 7.8 fits into the framework of [33] resp. Definition 8.3 if we choose  $A = (x_1)$ ,  $Y = (x_2, \dots, x_N)$  and  $\kappa$  as follows:  $\kappa(\text{span}(e_1)) := N - 2$  and on all other one-dimensional flats  $C$ ,  $\kappa(C) := 0$ . In this case, our space  $\mathcal{D}(X, \mathbb{B}')$  is the same as the space  $\mathcal{D}_\kappa$  in [33]. The  $\mathcal{P}$ -spaces are different because  $\kappa$  is not incremental.

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